

# Uncertainty Model Unfalsification With Simulation

Bruce R. Woodley <sup>1</sup>

Robert L. Kosut <sup>2</sup>

Jonathan P. How <sup>3</sup>

## Abstract

There has been much recent interest in model unfalsification for robust control of dynamic systems. In particular, it has been shown how to establish a tradeoff between sensor/process disturbances and linear time-invariant dynamic uncertainty using prediction error modeling. Unfalsification has also been integrated into adaptive control algorithms, in which a key component is the ability to trade between disturbances and dynamic uncertainty.

The main contribution presented here is the calculation of model unfalsification tradeoff curves for a simple simulated system. The unfalsified dynamic uncertainties are compared to the difference between the identified models and the true model.

## 1 Uncertainty Modeling

The starting point for robust control design is an *uncertainty model* which accounts for parametric, dynamic, and disturbance uncertainty. The specific form considered here, described in detail in [2], is referred to as a *prediction error (PE) uncertainty model*:

$$y = G_\theta u + H_\theta(w + \Delta u) \quad \begin{cases} \theta \in \Theta \\ \|w\|_{\text{rms}} \leq \sigma \\ \|\Delta\|_{\mathbf{H}_\infty} \leq \delta \end{cases} \quad (1)$$

where  $y$  and  $u$  are, respectively, the observed output and input sequences,  $w$  is an uncertain sequence RMS-bounded by  $\sigma$ ,  $\Delta$  is an uncertain transfer function  $\mathbf{H}_\infty$ -bounded by  $\delta$ ,  $G_\theta$  and  $H_\theta$  are causal, linear-time-invariant systems.

The uncertainty model (1) is said to be **unfalsified** by the  $N$ -point data sequences  $(y^N, u^N)$  if and only if there exists  $\theta$ ,  $\Delta(z)$ , and an  $N$ -point sequence  $w^N$ , such that:  $\theta \in \Theta$ ,  $\|\Delta\|_{\mathbf{H}_\infty} \leq \delta$ ,  $\|w^N\|_{\text{rms}} \leq \sigma$ , and (1) is consistent with the data, *i.e.*,

$$y^N = G_\theta u^N + H_\theta(w^N + \Delta u^N) \quad (2)$$

A data dependent test of unfalsification can be found in [2]. Using those results, the computation of an uncertainty tradeoff curve can be obtained by solving the fol-

lowing optimization problem:

$$\sigma(\delta) := \min_{\theta, \sigma, w^N} \sigma, \quad \text{subject to (1),(2)} \quad (3)$$

Denote the minimizing values of  $\theta$  and  $w^N$  by  $\theta(\delta)$  and  $w^N(\delta)$ .

If an ARX model is used,

$$\begin{aligned} G_\theta(z) &= \frac{B_\theta(z)}{A_\theta(z)}, & H_\theta(z) &= \frac{1}{A_\theta(z)} \\ A_\theta(z) &= 1 + a_1 z^{-1} + \dots + a_n z^{-n_A} \\ B_\theta(z) &= b_1 z^{-1} + \dots + b_n z^{-n_B} \\ \theta &= [a_1 \dots a_{n_A} \ b_1 \dots b_{n_B}]^T \in \mathbf{R}^p, \quad p = n_A + n_B \end{aligned}$$

then the tradeoff curve  $\sigma(\delta)$  is convex in  $\delta$ , and the optimizing values  $\theta(\delta)$  and  $w^N(\delta)$  are unique. Of practical importance is that the optimization problem (3) can be expressed as linear matrix inequalities (LMIs) in the unknown variables  $\delta \in \mathbf{R}_+$ ,  $\sigma \in \mathbf{R}_+$ ,  $\theta \in \mathbf{R}^p$ , and  $\text{vec}(w^N) \in \mathbf{R}^N$ , thus the problem can be solved very easily.

An approximate solution to the optimization problem (3), employing an ARX model is also given in [2]. The solution employs an iterative FFT technique that converges to the exact solution as the length of the observed input and output sequences increases ( $N \rightarrow \infty$ ).

## 2 System Simulations

The two identification procedures of problem (3) were applied to a discrete time ( $T = 0.25$ ) simulation of the following simple system

$$\ddot{x} + \dot{x} + x = u \quad (4)$$

The system was run for 15 seconds (61 time steps), with input

$$u_k = \begin{cases} 1, & k \in [0, 20] \\ 0, & k \in [21, 40] \\ 1, & k \in [41, 61] \end{cases} \quad (5)$$

Sensor noise was simulated as

$$y_{\text{meas}} = x + v \quad (6)$$

where  $v$  is zero mean, bandlimited white Gaussian noise, with  $\sigma = 0.01$ . The “exact” (LMI) and “approximate” (FFT) unfalsified tradeoff curves were calculated for the above data, using the ARX model, and assuming a second order system. The minimizations in problem (3) were calculated for  $\delta = \{0.0125, 0.0250, \dots, 0.2500\}$ .

<sup>1</sup>PhD Candidate, Department of Electrical Engineering, Stanford University, Stanford, California 94305, U.S.A. woodley@sun-valley.stanford.edu

<sup>2</sup>Vice President, Systems and Controls Division, SC Solutions, 3211 Scott Blvd. Santa Clara, California 95054, U.S.A. kosut@scsolutions.com

<sup>3</sup>Assistant Professor, Department of Aeronautics and Astronautics, Stanford University, Stanford, California 94305, U.S.A. howjo@sun-valley.stanford.edu

For the LMI solution, the calculations were performed using the Matlab LMI Toolbox. The approximate solution also utilized Matlab. The exact LMI solution took about 8 minutes for each value of  $\delta$ , whereas the approximate FFT solution took approximately 5 seconds for each value of  $\delta$ , both on an Ultra Sparc 1 workstation.

Because both the LMI and FFT solutions of (3) also return an estimate of plant model  $\hat{G}_\theta$ , it is possible to calculate  $(\delta_{true}, \sigma_{true})$  at each point along the tradeoff curve. That is, for each value of  $(\delta, \sigma)$ , calculate

$$\delta_{true} = \left\| \hat{H}_\theta^{-1}(\hat{G}_\theta - G_{true}) \right\|_{\mathbf{H}_\infty} \quad (7)$$

$$\sigma_{true} = \left\| \hat{H}_\theta^{-1}v \right\|_{\text{rms}} \quad (8)$$

In both cases, the  $\sigma_{true}$  vs.  $\delta_{true}$  curve appears above and to the right of the  $\sigma$  vs.  $\delta$  tradeoff curve. In the LMI case, this relationship is necessary, since the optimization of problem (3) is based on a finite data set, and will produce a  $w$  and  $\Delta$  which have been specialized to this data set. Thus  $\delta \leq \delta_{true}$ , and  $\sigma \leq \sigma_{true}$ . Because the FFT algorithm is approximate, no such guarantee exists. Thus it is significant that the approximate FFT technique produces  $\delta \leq \delta_{true}$ , and  $\sigma \leq \sigma_{true}$ . The four loci can be seen in Figure 1.

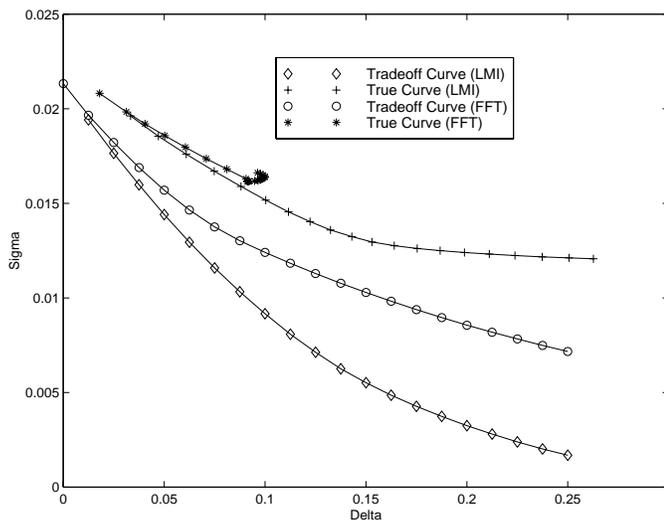


Figure 1: Uncertainty Tradeoff Curves

As an illustration of how the identified system models differ from the actual system, consider the model obtained when the LMI optimization is performed for  $\delta = \{0.0125, 0.0625, 0.2500\}$ . This is shown in Figure 2. These show the degradation in the ARX model fit as the size of the allowable uncertainty increases.

### 3 Discussion

The closeness of the identified model uncertainty to the actual model uncertainty using the LMI optimization in Figure 1 implies that this identification technique could be useful for the designing of robust controllers from identification data. The long calculation time, however,

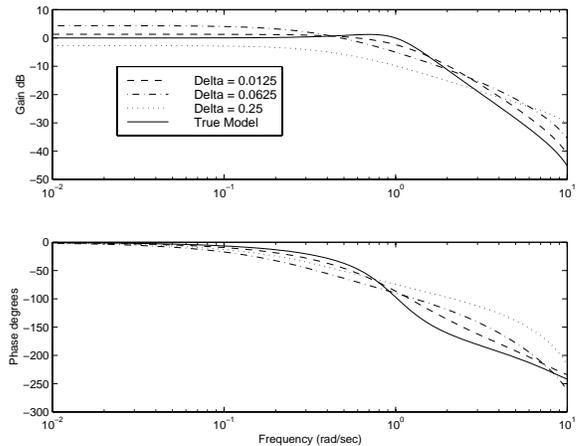


Figure 2: Bode Plot Comparison of Actual and Several Identified Models

makes this implementation impractical for real-time implementation on short data sets, and infeasible for long data sets. It is likely that significant improvements are possible if the LMI codes were optimized for this application.

The approximate techniques show promise in their ability to produce relatively fast results. Further investigation is required to determine if longer data sets will bring the approximate tradeoff curve closer to that of the exact LMI tradeoff curve.

### 4 Conclusions

If the goal of designing robust controllers for practical systems is to be achieved, a technique for identifying system models and appropriate system uncertainties must be developed. Uncertainty model unfalsification offers a promising solution to this problem.

This paper establishes the unfalsified tradeoff curve for a simple simulated system. A fast approximate technique for calculating this tradeoff curve was also investigated. Further numerical studies are required before this technique can be made feasible for application to a real-time system. Future work will extend these results to permit adaptive controller design.

### References

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