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# Parametric robust $H_2$ control design with generalized multipliers via LMI synthesis

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A new combined analysis and synthesis procedure that provides a less conservative robust control design technique for systems with real parametric uncertainty is presented. The robust stability for these systems is analysed by the passivity theorem with generalized multipliers, and the worst case $H_2$ performance is investigated using an upper bound on the total output energy. The dynamics of the multipliers are systematically chosen using knowledge from the linear part of the uncertain systems. This approach provides additional degrees of freedom in the synthesis that lead to a reduction of the conservatism in the worst-case $H_2$ performance and achieved robustness bounds. However, the formulation of the control design problem is very complicated and it is difficult to solve directly. This paper presents an iterative algorithm, which in an $H_2$ equivalent of the D–K iteration for the $\mu/K_m$ synthesis, to account for the complicated couplings in the synthesis problem. We use a simple beam system with an uncertain modal frequency to illustrate that this synthesis technique with generalized multipliers results in less conservative controllers than previously published Popov controller synthesis techniques. In the process, we demonstrate that this design approach is very effective and simple to implement numerically.

## 1. Introduction

Reducing conservatism in the robust stability analysis and synthesis for systems with real parametric uncertainty is currently a key issue in the controls community, but its heritage dates back to the 1960s. The primary focus of the early works (Zames 1996 a, b, Sandberg 1964) was not directly on the issue of the parametric uncertainty, but more on developing a fundamental understanding of the multivariable stability analysis based on the conic-sector, positivity, and loop gain concepts. This led more recently to quantitative measures such as the multivariable stability margin $K_m$ (Safonov 1982), and the structured singular value $\mu$ (Doyle 1982). This analysis was extended to the robust controller synthesis using variants of the so-called D–K iteration (Safonov 1983, Doyle 1983), which have recently been improved by including real parametric uncertainty (Fan *et al.* 1991, Young 1993) and by eliminating the need for curve-fitting (Safonov and Chiang 1993). The $\mu/K_m$ synthesis problem using Bilinear Matrix Inequalities (BMIs) was formulated by Safonov *et al.* (1994), Goh *et al.* (1994). The BMI approach has been shown to improve the guaranteed lower bounds of the multivariable stability margins by 10% over the corresponding results from the D–K iteration, with no increase in the controller order. A key advantage of the BMI technique is that it enables control engineers to address

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several other open problems of the robust control synthesis, namely fixed order control synthesis and decentralized controller architecture.

Balakrishnan (1995) presented a unified framework for robust stability tests based on the passivity theorem with multipliers and also demonstrated that these tests can be performed using convex optimization over Linear Matrix Inequalities (LMIs). Boyd *et al.* (1994), Feron (1994) and How (1993) considered the robust $H_2$ performance analysis for uncertain systems, and also focused on including these stability multipliers. Although the actual worst-case $H_2$ performance is very difficult to compute, its upper bound is much easier to calculate. These authors have demonstrated that the conservatism of the bound for the worst-case $H_2$ performance could be reduced by applying the passivity theorem with generalized multipliers, and that the formulation of the analysis tests naturally leads to LMIs. These robust analysis approaches have been tested for simple systems using generalized, dynamic multipliers. However, they have not been extended to the design of robust controllers that guarantee the robust stability and $H_2$ performance. Although the robust designs with Popov multipliers (or a slight generalization of Popov multipliers for repeated uncertainty) have been previously shown by several authors (Haddad and Bernstein 1991, How 1993, How *et al.* 1994, Sparks and Berstein 1995, Banjerdpongchai and How 1996), these controllers are designed for systems with sector bounded nonlinear uncertainty, which could be a source of conservatism when working with systems for real parametric uncertainty.

In this paper we introduce a new, combined analysis and synthesis procedure that leads to an effective robust control design technique, and we demonstrate that this technique can be used to design less conservative controllers for systems with real parametric uncertainty. Our approach is motivated by the work of Haddad *et al.* (1992) Balakrishnan (1995) and Feron (1994). The direct extension of the robust analysis to the controller synthesis results in BMIs, which currently are difficult to solve directly (Toker and Özbay 1995). As a result, El Ghaoui and Balakrishnan (1994) proposed an iterative procedure for solving BMI problems using a two-stage optimization process, called the V–K iteration. Some of the design variables in the BMIs are fixed during each phase of this iteration, leading to LMIs in the remaining variables. This technique has been shown to work well on simple examples, but on complicated objectives, such as robust control designs to minimize an $H_2$ performance, this approach has been found to converge very slowly if at all. This synthesis algorithm was recently improved leading to a systematic control design for systems with unstructured uncertainty (El Ghaoui and Folcher 1996). We have already successfully applied an extension of this algorithm to the parametric robust $H_2$ control design with Popov multipliers (Banjerdpongchai and How 1996). Key advantages of the LMI formulation when compared to the previously used gradient techniques are the simplicity and low overhead in the numerical implementation. In this paper we extend the synthesis to the case of generalized multipliers, which is an important step in developing less conservative controllers for systems with real parametric uncertainty.

Our design objective is achieved by combining the passivity analysis using multipliers and the worst-case $H_2$ performance, i.e. the output energy, of an LTI system subject to real parametric uncertainty. In the process we show the difficulties that arise when we simultaneously select the optimal parameters for both the multiplier and the compensator. We take advantage of the closed-loop matrix structure to eliminate some design parameters from the problem formulation using a simple algebraic technique. Although the problem size and the number of design parameters are reduced, some couplings still remain. Hence, we apply an iterative algorithm using LMI synthesis tools (Vandenberghe and Boyd 1994, Wu and Boyd 1996) to solve the design problem. As we will show, this approach is quite distinct from the D–K iteration for the $\mu/K_m$ synthesis because some variables are shared between the two main stages of the iterative solution.

The paper is organized as follows. In the next section we present the various mathematical notations and lemmas used in this paper. The robust controller synthesis is quite complex and the algorithm draws on a variety of techniques to address the multiplier selection as well as the controller parametrization. The synthesis formulation using these previously published techniques is developed in section 3. The solution procedure and algorithm are presented in section 4. Lastly, we use the Bernoulli Euler Beam system to demonstrate that the synthesis with generalized multipliers leads to less conservative controllers for systems with real parametric uncertainty.

## 2. Preliminaries

The following briefly summarizes the key notations which we will use to present the main theoretical results in sections 3.1 and 3.2.

$\mathbb{R}(\mathbb{C})$ denotes the set of real (complex) numbers. $\mathbb{R}_+$ denotes the set of non-negative real numbers. $\mathbb{R}^{m\times n}(\mathbb{C}^{m\times n})$ is the vector space of $m\times n$ real (complex) matrices. For any matrix $A\in\mathbb{R}^{m\times n}$, $A^{\mathrm{T}}$ denotes its transpose and for any matrix $A\in\mathbb{C}^{m\times n}$, $A^*$ denotes its complex conjugate transpose. $A_\perp$ denotes an orthogonal complement of $A$, i.e. $A^{\mathrm{T}}A_\perp=0$ and $[A\ A_\perp]$ is of maximum rank. The identity matrix is denoted by $I$. If $A\in\mathbb{R}^{n\times n}$, $\mathrm{Tr}\,A$ denotes the trace of $A$. If $A$ is square and invertible, then $A^{-1}$ is its inverse. Given a set of matrices $A_1\in\mathbb{R}^{n_1\times n_1},\dots,A_N\in\mathbb{R}^{n_N\times n_N}$, and $n=\sum_{i=1}^N n_i$, $\mathrm{diag}(A_1,\dots,A_N)$ denotes the $n\times n$ matrix

$$\begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & A_N \end{bmatrix}$$

When there is no ambiguity, $\mathrm{diag}_i(A_i)$ or $\mathrm{diag}_{i=1}^N(A_i)$ denotes $\mathrm{diag}(A_1,\dots,A_N)$. For any two matrices $A$ and $B\in\mathbb{R}^{n\times n}$, the inequality $A<B\,(A\le B)$ means that both $A$ and $B$ are symmetric and that $B-A$ is positive definite (positive semidefinite).

$\mathcal{L}_2^n$ is the Hilbert space of square-integrable signals defined over $\mathbb{R}_+$ with $n$ components, i.e. $u\in\mathcal{L}_2^n$ satisfying

$$\int_0^\infty u^{\mathrm{T}}u\,\mathrm{d}t<\infty$$

$\mathcal{L}_2^n$ is often abbreviated as $\mathcal{L}_2$. A causal $n$-input $n$-output operator $F:\mathbb{R}^n\to\mathbb{R}^n$ is said to be $\mathcal{L}_2$ stable if there exist $\gamma\ge 0$ and $\beta$ such that

$$\|Fu\|_2\le\gamma\|u\|_2+\beta,\quad\forall u\in\mathcal{L}_2 \tag{1}$$

where $\|\cdot\|_2$ is defined as the $\mathcal{L}_2$ norm. The $\mathcal{L}_2$ gain of $F$ is defined as the smallest $\gamma$ such that (1) holds for some $\beta$. For the linear operator $F$ we have the following definitions. $I$ is the identity operator; $F^{-1}$ is the inverse of $F$, i.e. $FF^{-1} = I$; $F^*$ is the adjoint of $F$; and $F^{-*}$ is the inverse of the adjoint of $F$. $F$ is said to be passive if

$$\int_0^T u(t)^{\mathrm{T}}(Fu)(t)\,\mathrm{d}t \geq 0, \quad \forall T \geq 0, \quad \forall u \in \mathcal{L}_2$$

It is strictly passive if it satisfies

$$\int_0^T u(t)^{\mathrm{T}}(Fu)(t)\,\mathrm{d}t > 0, \quad \forall T \geq 0, \quad \forall u \in \mathcal{L}_2$$

Let $F$ be an LTI system with a transfer function $F(s)$. Suppose that $F(s)$ is stable, i.e. all poles are on the open left half of the $s$-plane. In the frequency domain, the condition for $F(s)$ to be passive is that

$$F(j\omega) + F(j\omega)^* \geq 0, \quad \forall \omega \in \mathbb{R} \tag{2}$$

$F(s)$ is said to be positive real if $F(s)$ satisfies (2). $F(s)$ is strictly positive real if it satisfies

$$F(j\omega) + F(j\omega)^* > 0, \quad \forall \omega \in \mathbb{R}$$

The state-space condition for a passive LTI system is given in the following lemma. The form of the lemma (Anderson and Vongpanitlerd 1973, Chaps 5–7) is originally stated as matrix equations, but here we will use linear matrix inequality form.

**Lemma 2.1—Positive real lemma** (Boyd *et al.* 1994): *Let $F(s)$ be a transfer matrix of a stable LTI system, with the minimal realization $\{A, B, C, D\}$. $F(s)$ is positive real or passive if and only if there exists $P = P^{\mathrm{T}} > 0$ satisfying*

$$\begin{bmatrix} PA + A^{\mathrm{T}}P & PB - C^{\mathrm{T}} \\ B^{\mathrm{T}}P - C & -(D + D^{\mathrm{T}}) \end{bmatrix} \leq 0.$$

**Proof:** For the proof, see Boyd *et al.* (1994).

We note that $F(s)$ is strictly positive real or strictly passive if there exists $P = P^{\mathrm{T}} > 0$ satisfying

$$\begin{bmatrix} PA + A^{\mathrm{T}}P & PB - C^{\mathrm{T}} \\ B^{\mathrm{T}}P - C & -(D + D^{\mathrm{T}}) \end{bmatrix} < 0$$

The following lemmas will be very useful in developing the controller design technique in section 4.

**Lemma 2.2—Elimination lemma** (Boyd *et al.* 1994): *Let $G \in \mathbb{R}^{n \times n}$, $U \in \mathbb{R}^{n \times p}$ and $V \in \mathbb{R}^{n \times q}$. There exists a matrix $X \in \mathbb{R}^{p \times q}$ such that*

$$G + VX^{\mathrm{T}}U^{\mathrm{T}} + UXV^{\mathrm{T}} < 0$$

*if and only if*

$$V_{\perp}^{\mathrm{T}} G V_{\perp} < 0, \quad U_{\perp}^{\mathrm{T}} G U_{\perp} < 0$$

**Proof:** For the proof, see Boyd *et al.* (1994).

**Lemma 2.3—Completion lemma** (Packard *et al.* 1991): *Let $P$ and $Q \in \mathbb{R}^{m \times m}$ be positive definite matrices. There exists a positive definite matrix $P \in \mathbb{R}^{2m \times 2m}$ such that the upper left $m \times m$ block of $\tilde{P}$ is $P$, and that of $\tilde{P}^{-1}$ is $Q$, if and only if*

$$\begin{bmatrix} P & I \\ I & Q \end{bmatrix} \geq 0 \qquad (3)$$

**Proof:** For the proof, see Packard *et al.* (1991).

For each pair of matrices $P$ and $Q$ that strictly satisfy (3), the set of matrices $\tilde{P}$ satisfying the conditions in Lemma 2.3 is parametrized by

$$\tilde{P} = \begin{bmatrix} I & 0 \\ 0 & M^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} P & I \\ I & (P - Q^{-1})^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}$$

where $M \in \mathbb{R}^{m \times m}$ is an arbitrary invertible matrix. Then

$$\tilde{Q} = \tilde{P}^{-1} = \begin{bmatrix} I & 0 \\ 0 & N^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} Q & I \\ I & (Q - P^{-1})^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}$$

where $N = (I - QP)M^{-1}$.

## 3. Problem statement

We consider an LTI system, i.e. the nominal system $G$, subject to the uncertainty $\Delta$ (see figure 1, where $K$ is not considered) described by

$$\left. \begin{aligned} \dot{x} &= Ax + B_p p + B_w w + B_u u \\ q &= C_q x + D_{qp} p + D_{qw} w + D_{qu} u \\ z &= C_z x + D_{zp} p + D_{zw} w + D_{zu} u \\ y &= C_y x + D_{yp} p + D_{yw} w + D_{yu} u \\ p &= -\Delta q \end{aligned} \right\} \qquad (4)$$

where $x : \mathbb{R}_+ \to \mathbb{R}^n$ is the state, $u : \mathbb{R}_+ \to \mathbb{R}^{n_u}$ is the control input, $w : \mathbb{R}_+ \to \mathbb{R}^{n_w}$ is the disturbance input, $y : \mathbb{R}_+ \to \mathbb{R}^{n_y}$ is the mesured output, $z : \mathbb{R}_+ \to \mathbb{R}^{n_z}$ is the performance output, $q : \mathbb{R}_+ \to \mathbb{R}^{n_p}$ and $p : \mathbb{R}_+ \to \mathbb{R}^{n_p}$ are the input/output of the uncertainty $\Delta$. The uncertainty $\Delta$ is assumed to be a diagonal constant matrix with positive elements, i.e. $\Delta \in \boldsymbol{\Delta}$ where

$$\boldsymbol{\Delta} := \{\Delta : \Delta = \mathrm{diag}(\delta_1, \ldots, \delta_{n_p}), \quad \text{and} \quad \delta_i > 0, \quad \forall i = 1, \ldots, n_p\}$$

In control theory, this is referred to as real parametric uncertainty.

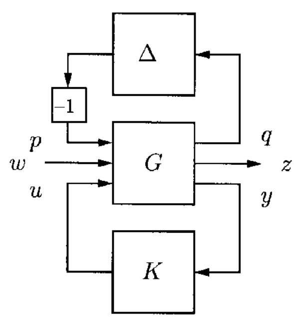


Figure 1. Elements of the robust synthesis problem.

For well-posedness, $D_{zw}$ is assumed to be identically zero. To significantly simplify the analysis and synthesis, we assume that $D_{zp}$, $D_{qw}$ and $D_{qu}$ are identically zero.

**Remark 3.1:** We note that this formulation can easily be extended to handle the system with the constant diagonal uncertainty $\Delta$ with elements satisfying $|\delta_i| < \gamma$, $\forall i = 1, \ldots, n_p$. In these cases we apply a bilinear sector transformation (Desoer and Vidyasagar 1975) so that the stability of the uncertain system can be analysed by the passivity theorem. In such cases, we assume that $(\gamma I + \Delta)$ is invertible (Desoer and Vidyasagar 1975) and define $G_{qp}(s)$ to be the transfer matrix from $p$ to $q$. After the transformation the uncertain system (4) is described by

$$\tilde{G}_{qp}(s) = (I - \gamma G_{qp}(s))^{-1}(I + \gamma G_{qp}(s))$$
$$\tilde{\Delta} = (\gamma I - \Delta) \circ (\gamma I + \Delta)^{-1}$$

where $\circ$ denotes a composition operator. As discussed in Anderson (1972), it can be shown that $\Delta$ has an $\mathcal{L}_2$ gain less than $\gamma$ (i.e. $|\delta_i| < \gamma$) if and only if $\tilde{\Delta}$ is strictly passive (i.e. $\tilde{\delta}_i > 0$). Given $G_{qp}$ with a state-space realization $\{A, B_p, C_q, D_{qp}\}$, the state-space realization of $\tilde{G}_{qp}$ is

$$\{A + \gamma B_p(I - \gamma D_{qp})^{-1}, 2B_p(I - \gamma D_{qp})^{-1}, (I - \gamma D_{qp})^{-1}\gamma C_q, (I + \gamma D_{qp})(I - \gamma D_{qp})^{-1}\}$$

Other classes of the uncertainty, such as a diagonal passive operator or a diagonal passive LTI uncertainty, can be handled in a similar manner (Balakrishnan 1995). An extension to the classes of uncertainty with elements having an $\mathcal{L}_2$ gain less than $\gamma$ is also straightforward via the bilinear sector transformation.

The objective of this paper is to design a strictly proper full-order LTI controller using multiplier theory for the uncertain system (4) such that the robust stability of the system is achieved and an upper bound of the worst-case $H_2$ performance is minimized. The formulation is quite complicated because it requires a simultaneous selection of the optimal parameters of both the multipliers and compensators. In the following subsections we will develop the robust $H_2$ synthesis formulation from the robust $H_2$ analysis with multipliers.

### 3.1. *Absolute stability analysis with multipliers*

The robust stability analysis is based on the passivity theorem with multipliers, i.e. multiplier theory (Zames 1966 a, b, Desoer and Vidyasagar 1975). These multipliers are devised to capture additional information, i.e. structure and type, of the uncertainty $\Delta$ in order to obtain less conservative conditions for robustness analysis. The application of multiplier theory is given in the following theorem. The benefits of including this additional information will be discussed in the numerical example section 5.

**Theorem 3.1** (Desoer and Vidyasagar 1975): *Consider the system* (4) *and assume that it has a solution* $x \in \mathcal{L}_2$. *Let* $u = 0$ *and* $w = 0$. *Suppose that there exists an operator* $W : \mathcal{L}_2 \to \mathcal{L}_2$ *satisfying the following conditions.*

(1) *W can be factored such that* $W = W_- W_+$, *where* $W_-$, $W_+$, *and their inverses map* $\mathcal{L}_2$ *to* $\mathcal{L}_2$.

(2) $W_-$ *is linear, hence* $W_-^*$ *and* $W_-^{-*}$ *are well-defined.*
(3) $W$, $W_+$, $W_-^*$, *and their inverses have finite gains.*
(4) $W_+$, $W_-^*$ *and their inverses are causal.*

*If* $WG_{qp}$ *has a finite gain,* $WG_{qp}$ *is strictly passive, and* $\Delta W^{-1}$ *is passive, then the uncertain system (4) is* $\mathcal{L}_2$ *stable.*

**Proof:** For the proof, see Desoer and Vidyasagar (1975).

$W$ is called the stability multiplier. Because $W_-$ and its inverse are anticausal, and $W_+$ and its inverse are causal (Desoer and Vidyasagar 1975, Balakrishnan 1995), the multipliers satisfying the conditions in Theorem 3.1 are noncausal. Furthermore, because $\Delta$ and $G_{qp}$ are causal and $W$ is non-causal, $WG_{qp}$ and $\Delta W^{-1}$ are non-causal. Feron (1994) discusses a sufficient condition with many theoretical steps to search for a non-causal multiplier $W$ such that $WG_{qp}$ is strictly passive and the performance bound is achieved. This approach results in a sophisticated optimization problem over LMIs. However, Desoer and Vidyasagar (1975) show that for the multipliers satisfying the conditions in Theorem 3.1, $WG_{qp}$ (i.e. a non-causal operator) is strictly passive if and only if $W_+ G_{qp} W_-^{-*}$ (i.e. a causal operator) is strictly passive. Moreover, $\Delta W^{-1}$ is passive if and only if $W_-^* \Delta W_+^{-1}$ is passive. Balakrishnan (1997) considers the robust analysis of the general framework, i.e. the operator $W_+ G_{qp} W_-^{-*}$, and formulates the test as a convex optimization over LMI constraints. The underlying numerical methods are based on a state-space approach and they result in guaranteed performance bounds. This general framework might offer an alternative way to design robust controllers. However, the conservatism of these bounds obtained by this general framework remains for further investigation. For simplicity in the following synthesis formulation, we set $W_-$ equal to identity, so that $W = W_+$ (i.e. the set of causal multipliers). Although this choice of multipliers is more restricted than the generalized (non-causal) multipliers, this choice allows far more freedom than the Popov multipliers that have been investigated previously (Banjerdpongchai and How 1996). Future research will explore the advantages of using the factorized form of the generalized multipliers in the robust analysis/synthesis.

In practice, a finite dimensional approximation of the set of multipliers is used to test the assumptions in Theorem 3.1 (Safonov and Chiang 1993, Balakrishnan 1995, Feron 1994, Balakrishnan 1997). To capture real parametric uncertainty $\Delta \in \boldsymbol{\Delta}$, we select the multiplier $W$ which has the form

$$W = \begin{cases} \operatorname{diag}(W_1, \ldots, W_{n_p}) \\ \text{where } W_i \text{ is linear time-invariant, finite-dimensional,} \\ \text{stable, positive real and has no poles on the imaginary axis.} \end{cases} \qquad (5)$$

Note that the freedom in selecting this multiplier serves as a qualitative measure of the conservatism of the robustness test. The multipliers of the form (5) include very general parametrizations of the stability multipliers involving RL (resistor-inductor), RC (resistor-capacitor), and shifted LC (inductor-capacitor) (How and Haddad 1994). We denote the state-space realization of $W_i$ and $\{A_{W_i}, B_{W_i}, C_{W_i}, D_{W_i}\}$, then the multiplier $W$ is described by

$$\left.\begin{aligned} \dot{x}_W &= A_W x_W + B_W q \\ q_W &= C_W x_W + D_W q \end{aligned}\right\} \tag{6}$$

where $x_W : \mathbb{R}_+ \to \mathbb{R}^{n_W}$ is the multiplier state, $x_{W,0} = 0$ and

$$A_W = \mathrm{diag}_{i=1}^{n_p}(A_{W_i}), \quad B_W = \mathrm{diag}_{i=1}^{n_p}(B_{W_i})$$
$$C_W = \mathrm{diag}_{i=1}^{n_p}(C_{W_i}), \quad D_W = \mathrm{diag}_{i=1}^{n_p}(D_{W_i})$$

**Remark 3.2:** Other types of the uncertainty can be handled by choosing appropriate classes of multipliers (Balakrishnan 1995). For example, a constant diagonal positive definite matrix is chosen for the uncertainty that is a diagonal passive (linear or nonlinear) operator. For diagonal passive LTI uncertainty, multipliers are chosen to be real rational, diagonal, bounded on the imaginary axis, and so that they satisfy

$$W(j\omega) = W(j\omega)^* > 0, \quad \forall \omega \in \mathbb{R}$$

The next subsection closely parallels the developments in Feron (1994). We will focus on the worst-case $H_2$ performance of the uncertain system (4). This analysis will be used to formulate the controller synthesis problem in section 3.4.

### 3.2. *Worst-case* H$_2$ *performance*

The robust performance analysis forms the foundation of the robust controller synthesis presented in this paper. This subsection provides a brief overview of the performance analysis involving generalized multipliers (Feron 1994).

Consider the uncertain system (4). Let $x_0$ be any initial condition with zero disturbance input. Assume that the uncertain system (4) is stable. We are interested in computing the worst case output energy for the system subject to real parametric uncertainty, i.e. $J_{x_0} := \max_{\Delta \in \boldsymbol{\Delta}} \|z\|_2^2$. Although this quantity is very difficult to compute, we are interested in an upper bound which can be calculated relatively easily. The following lemma gives an upper bound on $J_{x_0}$.

**Lemma 3.1** (Feron 1994): *Consider the uncertain system* (4). *Let* $\mathcal{W}$ *be a family of multipliers which have the form* (5) *and satisfy the assumptions of Theorem* 3.1. *Then the output energy* $J_{x_0}$ *of the uncertain system* (4) *is bounded by*

$$J_{x_0} \le \min_{W \in \mathcal{W}} \max_{\hat{p} \in \mathcal{L}_2} \left( \|\hat{z}\|_2^2 - 2\int_0^\infty \hat{p}^{\mathrm{T}} \hat{q}_W \, \mathrm{d}t \right)$$

*where* $\hat{z}$, $\hat{p}$ *and* $\hat{q}_W$ *satisfy*

$$\left.\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}_p \hat{p} + \hat{B}_w w + \hat{B}_u u \\ \hat{q}_w &= \hat{C}_q \hat{x} + \hat{D}_{qp}\hat{p} + \hat{D}_{qw} w + \hat{D}_{qu} u \\ \hat{z} &= \hat{C}_z \hat{x} + \hat{D}_{zp}\hat{p} + \hat{D}_{zw} w + \hat{D}_{zu} u \\ \hat{y} &= \hat{C}_y \hat{x} + \hat{D}_{yp}\hat{p} + \hat{D}_{yw} w + \hat{D}_{yu} u \end{aligned}\right\} \tag{7}$$

*where* $\hat{x}^T = \left[x^{\mathrm{T}} \; x_W^{\mathrm{T}}\right]$, $\hat{x}_0^{\mathrm{T}} = \left[x_0^{\mathrm{T}} \; 0\right]$ *and*

$$\left[\begin{array}{c|c|c|c} \hat{A} & \hat{B}_p & \hat{B}_w & \hat{B}_u \\ \hline \hat{C}_q & \hat{D}_{qp} & \hat{D}_{qw} & \hat{D}_{qu} \\ \hline \hat{C}_z & \hat{D}_{zp} & \hat{D}_{zw} & \hat{D}_{zu} \\ \hline \hat{C}_y & \hat{D}_{yp} & \hat{D}_{yw} & \hat{D}_{yu} \end{array}\right] = \left[\begin{array}{cc|c|c|c} A & 0 & B_p & B_w & B_u \\ B_W C_q & A_W & B_W D_{qp} & 0 & 0 \\ \hline D_W C_q & C_W & D_W D_{qp} & 0 & 0 \\ \hline C_z & 0 & 0 & 0 & D_{zu} \\ \hline C_y & 0 & D_{yp} & D_{yw} & D_{yu} \end{array}\right]$$

**Proof:** For the proof, see Feron (1994).

We note that the term

$$\max_{\hat{p}\in\mathcal{L}_2}\left(-\int_0^\infty \hat{p}^{\mathrm{T}}\hat{q}_W\,\mathrm{d}t\right) \tag{8}$$

subject to (7) has the interpretation of the maximum extractable energy from $\hat{p}$ to $\hat{q}_W$ in state $\hat{x}_0$. Because of the relationship $\hat{p} = -\Delta\hat{q}$, where $\Delta$ is strictly passive, $W$ is passive, (8) has a non-negative value. Given $W \in \mathcal{W}$, computing the upper bound of the output energy is equivalent to computing

$$\max_{\hat{p}\in\mathcal{L}_2}\int_0^\infty \begin{bmatrix}\hat{x}\\ \hat{p}\end{bmatrix}^{\mathrm{T}} \begin{bmatrix}\hat{C}_z^{\mathrm{T}}\hat{C}_z & -\hat{C}_q^{\mathrm{T}} \\ -\hat{C}_q & -(\hat{D}_{qp}+\hat{D}_{qp}^{\mathrm{T}})\end{bmatrix} \begin{bmatrix}\hat{x}\\ \hat{p}\end{bmatrix}\mathrm{d}t \tag{9}$$

where $\hat{x}$ and $\hat{p}$ satisfy (7). Therefore, computing the bound is simply a linear quadratic optimal control problem which we will state in the following theorem. The solution could be obtained by standard methods (Willems 1971, Anderson and Moore 1990).

**Theorem 3.2** (Feron 1994): *Consider the uncertain system* (4). *Suppose that there exists a multiplier* $W$ *of the form* (5) *satisfying the assumptions of Theorem* 3.1. *Then the upper bound of the output energy is finite and can be computed as the optimization problem of minimizing* $\hat{x}_0\hat{P}\hat{x}_0$ *over the variables* $\hat{P}$, $P_{W_i}$, $C_{W_i}$ *and* $D_{W_i}$ *subject to*

$$\left.\begin{aligned} &\hat{P} = \hat{P}^{\mathrm{T}} > 0 \\ &\begin{bmatrix}\hat{A}^{\mathrm{T}}\hat{P}+\hat{P}\hat{A}+\hat{C}_z^{\mathrm{T}}\hat{C}_z & \hat{P}\hat{B}_p-\hat{C}_q^{\mathrm{T}} \\ \hat{B}_p^{\mathrm{T}}\hat{P}-\hat{C}_q & -(\hat{D}_{qp}+\hat{D}_{qp}^{\mathrm{T}})\end{bmatrix} < 0 \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} &\begin{bmatrix}A_{W_i}^{\mathrm{T}}P_{W_i}+P_{W_i}A_{W_i} & P_{W_i}B_{W_i}-C_{W_i}^{\mathrm{T}} \\ B_{W_i}^{\mathrm{T}}P_{W_i}-C_{W_i} & -(D_{W_i}+D_{W_i}^{\mathrm{T}})\end{bmatrix} \le 0 \\ &P_{W_i} = P_{W_i}^{\mathrm{T}} > 0, \quad \forall i \in 1,\dots,n_p \end{aligned}\right\} \tag{11}$$

**Proof:** For the proof, see Feron (1994).

Note that the condition (10) implies the strictly positive real constraint of $WG_{qp}$ and the condition (11) is equivalent to the positive real constraint of the multiplier $W$. Both (10) and (11) are LMIs in the variables $\hat{P}$, $P_{W_i}$, $C_{W_i}$ and $D_{W_i}$. For notational convenience, we define $P_W := \mathrm{diag}_{i=1}^{n_p}(P_{W_i})$.

For the uncertain system (4), the $H_2$ performance is derived from the total energy of the performance outputs $z_i(t)$ subject to impulse disturbances $\delta w_i$, $i = 1,\dots,n_w$. Consequently, we define the worst-case $H_2$ performance $J$

$$J := \max_{\Delta \in \boldsymbol{\Delta}} \sum_{i=1}^{n_w} \|z_i\|_2^2 \qquad (12)$$

Although $J$ is difficult to compute, its upper bound can be easily computed. As discussed by Stoorvogel (1993) the appropriate initial conditions $\hat{x}_i(0)$, $i = 1, \ldots, n_w$, used in the computation of this upper bound are equal to $\hat{B}_w e_i$, where $\{e_i\}_{i=1}^{n_w}$ is an orthonormal basis of the input disturbance space $\mathbb{R}^{n_w}$. By applying Theorem 3.2 to the worst-case $H_2$ performance in (12), $J$ is bounded by

$$J \leq \operatorname{Tr} \hat{B}_w^{\mathrm{T}} \hat{P} \hat{B}_w \qquad (13)$$

In summary, to compute the minimum cost overbound of the output energy for the uncertain system (4), we solve the following optimization problem which we refer to as the worst-case $H_2$ performance analysis with multipliers.

$$\left.\begin{array}{ll} \text{minimize} & \operatorname{Tr} \hat{B}_w^T \hat{P} \hat{B}_w \\ \text{subject to} & (10) \text{ and } (11) \end{array}\right\} \qquad (14)$$

**Remark 3.3:** Although the analysis of the worst-case $H_2$ performance in this subsection is parallel to the previous technique of Feron (1994) a major difference exists. Feron (1994) gave an analysis involving a non-causal multiplier, which makes the initial condition of the multiplier state not equal to zero but depends on the input $\hat{p}$ in (7). Hence, in order to compute the uper bound of the output energy (9) a multiple-step technique is required, which subsequently complicates the optimization procedure. In contrast, in this work we use the causal multiplier of the form (5), which results in a zero initial condition of the multiplier state, i.e. $x_{W,0} = 0$. As we will show in section 3.4, the analysis formulation can easily be extended to the synthesis problem, which results in a clean and concise presentation.

In the next subsection we will present a systematic way of choosing a basis of the multiplier by using the information of the open-loop transfer function (the linear time-invariant part) of the uncertain system (4).

### 3.3. *Multiplier construction*

In previous approaches (Safonov and Chiang 1993, Balakrishnan 1995, Feron 1994), arbitrary basis functions for the multiplier are chosen without taking into account the knowledge of the open-loop transfer function of the uncertain system (4). However, Brockett and Willems (1965) provide an explicit expression for the multipliers that make $WG_{qp}$ strictly passive for the case of a single uncertainty. Their approach uses the information of the open-loop transfer function of the uncertain system to place poles of the multiplier, which is referred to as a plant–dependent multiplier. Brockett and Willems (1965) gave a multiplier of the form

$$\tau + s^{\pm 1} \frac{\prod_i (s^2 + \alpha_i^2)}{\prod_j (s^2 + \beta_j^2)}$$

where $\tau > 0$, $\alpha_i$ and $\beta_j$ are the frequencies satisfying

$$\left.\begin{array}{l} \arg\left[G_{qp}(\mathrm{j}\omega)\right] = 0 \\ \dfrac{\mathrm{d}\arg\left[G_{qp}(\mathrm{j}\omega)\right]}{\mathrm{d}\omega} \begin{cases} < 0, & \omega = \alpha_i \\ > 0, & \omega = \beta_j \end{cases} \end{array}\right\} \tag{15}$$

where $\arg\left[G_{qp}(\mathrm{j}\omega)\right]$ is the phase of $G_{qp}(\mathrm{j}\omega)$. In this paper we consider a family of multipliers for which $(A_W, B_W)$ are fixed, and $(C_W, D_W)$ are free to vary, provided that the conditions in Theorem 3.1 are satisfied. The reason for this restriction will become clear when we implement the robust performance analysis test. As suggested by Brockett and Willems (1965) and How and Haddad (1994) we will place the eigenvalues of $A_W$ to the locations that have the natural frequencies equal to $\beta_j$ satisfying (15). In order to make the multiplier contain no poles on the imaginary axis, we slightly shift the eigenvalues of $A_W$ into the left half of the $s$-plane by adding an arbitrary small damping ratio $\zeta_{W,j}$. As the state-space realization of the multiplier is arbitrary, we use a modal realization which automatically gives $B_W$ within a constant. This idea of choosing $(A_W, B_W)$ can be extended to the multiple uncertainty case where $(A_{W_i}, B_{W_i})$ are constructed by the information of the $(i, i)$ element of $G_{qp}$.

In the next subsection we will use the analysis tool presented in the preceding section to design an LTI controller such that the performance objective is satisfied.

### 3.4. *Multiplier controller synthesis*

This paper shows, for the first time, the extension of the robustness tests with generalized multipliers to the robust $H_2$ synthesis. The robust $H_2$ performance analysis can be used to design robust controllers. The design objective is to find a strictly proper full order LTI controller that minimizes the upper bound of the worst-case $H_2$ performance. The controller is of the form

$$\left.\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c \end{aligned}\right\} \tag{16}$$

where $x_c : \mathbb{R}_+ \to \mathbb{R}^{n+n_W}$ is the controller state and $(A_c, B_c, C_c)$ are constant matrices of appropriate size. We proceed by first describing the closed-loop system of the augmented system (7) and the LTI controller (16) by

$$\left.\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}_p\hat{p} + \tilde{B}_w w \\ \hat{q}_W &= \tilde{C}_q\tilde{x} + \tilde{D}_{qp}\hat{p} + \tilde{D}_{qw} w \\ \hat{z} &= \tilde{C}_z\tilde{x} + \tilde{D}_{zp}\hat{p} + \tilde{D}_{zw} w \end{aligned}\right\} \tag{17}$$

where $\tilde{x}^{\mathrm{T}} = \left[\hat{x}^{\mathrm{T}} \;\; x_c^{\mathrm{T}}\right]$ and

$$\left[\begin{array}{c|c|c} \tilde{A} & \tilde{B}_p & \tilde{B}_w \\ \hline \tilde{C}_q & \tilde{D}_{qp} & \tilde{D}_{qw} \\ \hline \tilde{C}_z & \tilde{D}_{zp} & \tilde{D}_{zw} \end{array}\right] = \left[\begin{array}{cc|c|c} \hat{A} & \hat{B}_u C_c & \hat{B}_p & \hat{B}_w \\ B_c\hat{C}_y & A_c + B_c\hat{D}_{yu}C_c & B_c\hat{D}_{yp} & B_c\hat{D}_{yw} \\ \hline \hat{C}_q & \hat{D}_{qu}C_c & \hat{D}_{qp} & 0 \\ \hline \hat{C}_z & \hat{D}_{zu}C_c & 0 & 0 \end{array}\right]$$

Then, it is straightforward to compute the upper bound of the worst-case $H_2$ performance for the closed-loop system (17). We note that the condition (10) is equivalent to $\tilde{P} = \tilde{P}^{\mathrm{T}} > 0$, and

$$\begin{bmatrix} \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} + \tilde{C}_z^T\tilde{C}_z & \tilde{P}\tilde{B}_p - \tilde{C}_q^T \\ \tilde{B}_p^T\tilde{P} - \tilde{C}_q & -(\tilde{D}_{qp} + \tilde{D}_{qp}^T) \end{bmatrix} < 0 \tag{18}$$

In summary, the design objective is to solve the following optimization problem:

$$\left.\begin{array}{ll} \text{minimize} & \operatorname{Tr} \tilde{B}_w^T\tilde{P}\tilde{B}_w \\ \text{subject to} & (11),(18),\tilde{P} > 0 \end{array}\right\} \tag{19}$$

## 4. Design procedure

We first note that (19) is a BMI problem, i.e. there are product terms involving the analysis parameters ($\tilde{P}, C_W$ and $D_W$) and compensator parameters ($A_c, B_c$ and $C_c$). The formulation is quite complicated because it requires a simultaneous optimization of both the multipliers and compensators. Observing the structure of the compensator parameters in (19), the first step of the design procedure is to eliminate some controller parameters from the problem formulation. We then solve for the remaining variables, and use these results to construct the controllers. An iterative algorithm is required to calculate the compensators, but this process capitalizes on the very efficient design tools that are available for solving LMI problems (Vandenberghe and Boyd 1994, Wu and Boyd 1996).

### 4.1. *Controller elimination*

We first note that the controller matrix $A_c$ only appears in (18). Thus it is possible to reduce the number of variables in the optimization problem by eliminating $A_c$. To proceed, we define

$$\tilde{A}_0 := \begin{bmatrix} \hat{A} & \hat{B}_u C_c \\ B_c\hat{C}_y & B_c\hat{D}_{yu}C_c \end{bmatrix}, \quad \tilde{J} := \begin{bmatrix} 0 \\ I \end{bmatrix}$$

Then $\tilde{A}$ can be written as $\tilde{A} = \tilde{A}_0 + \tilde{J}A_c\tilde{J}^{\mathrm{T}}$ and we rewrite (18) as

$$\tilde{G} + VA_c^{\mathrm{T}}U^{\mathrm{T}} + UA_cV^{\mathrm{T}} < 0 \tag{20}$$

where $\tilde{G}$, $U$ and $V$ are defined as

$$\tilde{G} := \begin{bmatrix} \tilde{A}_0^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A}_0 + \tilde{C}_z^{\mathrm{T}}\tilde{C}_z & \tilde{P}\tilde{B}_p - \tilde{C}_q^{\mathrm{T}} \\ \tilde{B}_p^{\mathrm{T}}\tilde{P} - \tilde{C}_q & -(\tilde{D}_{qp} + \tilde{D}_{qp}^{\mathrm{T}}) \end{bmatrix}$$

$$U := \begin{bmatrix} \tilde{P}\tilde{J} \\ 0 \end{bmatrix}, \quad V := \begin{bmatrix} \tilde{J} \\ 0 \end{bmatrix}$$

Therefore, the orthogonal complements of $U$ and $V$ are

$$U_{\perp} = \begin{bmatrix} \tilde{P}^{-1}\tilde{J}_{\perp} & 0 \\ 0 & I \end{bmatrix}, \quad V_{\perp} = \begin{bmatrix} \tilde{J}_{\perp} & 0 \\ 0 & I \end{bmatrix}$$

By applying the elimination lemma, it follows that (20) holds if and only if

$$\begin{bmatrix} \tilde{J}_{\perp} & 0 \\ 0 & I \end{bmatrix}^{\mathrm{T}} \tilde{G} \begin{bmatrix} \tilde{J}_{\perp} & 0 \\ 0 & I \end{bmatrix} < 0, \quad \begin{bmatrix} \tilde{P}^{-1}\tilde{J}_{\perp} & 0 \\ 0 & I \end{bmatrix}^{\mathrm{T}} \tilde{G} \begin{bmatrix} \tilde{P}^{-1}\tilde{J}_{\perp} & 0 \\ 0 & I \end{bmatrix} < 0 \tag{21}$$

To proceed, we partition $\tilde{P}$ and its inverse $\tilde{Q}$ as

$$\tilde{P} = \begin{bmatrix} P & M \\ M^T & R \end{bmatrix}, \quad \tilde{Q} = \tilde{P}^{-1} = \begin{bmatrix} Q & N \\ N^T & S \end{bmatrix} \tag{22}$$

where $P$ and $Q \in \mathbb{R}^{(n+n_W)\times(n+n_W)}$. Then, after some algebra, it can be shown that (21) are equivalent to

$$\left.\begin{aligned} &\begin{bmatrix} P\hat{A} + Z\hat{C}_y + (P\hat{A} + Z\hat{C}_y)^{\mathrm{T}} + \hat{C}_z^{\mathrm{T}}\hat{C}_z & P\hat{B}_p + Z\hat{D}_{yp} - \hat{C}_q^{\mathrm{T}} \\ (P\hat{B}_p + Z\hat{D}_{yp} - \hat{C}_q^{\mathrm{T}})^{\mathrm{T}} & -(\hat{D}_{qp} + \hat{D}_{qp}^{\mathrm{T}}) \end{bmatrix} < 0 \\ &\begin{bmatrix} \hat{A}Q + \hat{B}_u Y + (\hat{A}Q + \hat{B}_u Y)^{\mathrm{T}} & \hat{B}_p - Q^{\mathrm{T}}\hat{C}_q^{\mathrm{T}} & (\hat{C}_z Q + \hat{D}_{zu} Y)^{\mathrm{T}} \\ (\hat{B}_p - Q\hat{C}_q^{\mathrm{T}})^{\mathrm{T}} & -(\hat{D}_{qp} + \hat{D}_{qp}^{\mathrm{T}}) & 0 \\ \hat{C}_z Q + \hat{D}_{zu} Y & 0 & -I \end{bmatrix} < 0 \end{aligned}\right\} \tag{23}$$

where $Y$ and $Z$ are defined as $Y := C_c N^{\mathrm{T}}$, $Z := MB_c$. By the completion lemma, the conditions $\tilde{P} > 0$, $\tilde{P}\tilde{Q} = I$ with $\tilde{P}$ given by (22) imply

$$\begin{bmatrix} P & I \\ I & Q \end{bmatrix} \geq 0 \tag{24}$$

Restricting (24) to be positive definite, we are effectively searching for full-order controllers, i.e. of order $n + n_W$ (El Ghaoui and Folcher 1996). We observe that the second inequality in (23) is BMI, i.e. there are product terms involving $Q$ and $(C_W, D_W)$. This is a direct consequence of optimizing both the compensator parameters (related to $Q$) and the analysis multiplier $(C_W, D_W)$ simultaneously. Note that if $(C_W, D_W)$ are fixed, then (23) are LMIs in $Q$. Similarly, if $Q$ is fixed then (23) are LMIs in $(C_W, D_W)$.

Now we consider the objective function which is an upper bound of the worst-case $H_2$ performance, i.e. $\operatorname{Tr} \tilde{B}_w^{\mathrm{T}}\tilde{P}\tilde{B}_w$. The trace objective is equivalent to

$$\operatorname{Tr} \begin{bmatrix} \hat{B}_w \\ \hat{D}_{yw} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P & Z \\ Z^{\mathrm{T}} & X \end{bmatrix} \begin{bmatrix} \hat{B}_w \\ \hat{D}_{yw} \end{bmatrix} \tag{25}$$

and the existence of a symmetric matrix $X$ such that

$$\begin{bmatrix} X & Z^{\mathrm{T}} & 0 \\ Z & P & I \\ 0 & I & Q \end{bmatrix} > 0 \tag{26}$$

Following El Ghaoui and Folcher (1996), we note that (26) implies $\tilde{P} > 0$. In summary, after eliminating $A_c$ from the formulation the optimization problem (19) is equivalent to

$$\left.\begin{aligned} &\text{minimize} && (25) \\ &\text{subject to} && (11),(23),(26) \end{aligned}\right\} \tag{27}$$

## 4.2. *Controller reconstruction*

Given that there exist $P$, $Q$, $Y$, $Z$, $X$, $P_W$, $C_W$ and $D_W$ satisfying (27), we can construct a controller using the following procedure. We first construct $\tilde{P}$ such that (18) holds. $\tilde{P}$ is parametrized by (22), where $M$ is an arbitrary invertible matrix. Because $M$ corresponds to a change of coordinates in the controller states $x_c$, the choice of $M$ has no effect on the controller transfer function (El Ghaoui and Folcher 1996). After constructing $\tilde{P}$, the set of input/output controller matrices $(B_c, C_c)$ can be parametrized by $B_c = M^{-1}Z$ and $C_c = Y(I - PQ)^{-1}M$. With $\tilde{P}$, $C_W$, $D_W$, $B_c$ and $C_c$ determined, it suffices to find $A_c$ that satisfies (20), which can then be formulated as an LMI problem in $A_c$.

### 4.3. *Algorithm*

It has already been shown that BMI problems are NP-hard, and it is thought to be rather unlikely that there is a polynomial time algorithm to solve the general BMI problem (Toker and Özbay 1995). As there are product terms involving compensator parameters and the multiplier parameters, our approach to solve the non-convex optimization problem is based on an iterative procedure. First, we systematically select the multiplier dynamics $(A_W, B_W)$ by the method described in section 3.3. The proposed algorithm, which we call the V–K iteration, is basically an iteration between three different LMI problems, i.e. (19) with fixed compensator parameters, (27) with fixed multiplier parameters, and (20) over $A_c$. The first LMI problem, considered as the $V$ or analysis step, is to solve (19) with fixed $(A_c, B_c,$ and $C_c)$ which yields multiplier parameters ($C_W$ and $D_W$). For the $K$ or synthesis step, the second and third LMI problems are solved. The solution parameters of the second LMI problem, i.e. (27) with fixed multiplier parameters, implicitly contains the input/output compensator matrices ($B_c$ and $C_c$) as variables. After obtaining $B_c$ and $C_c$, the dynamics of the compensator $A_c$ can be computed by solving the third LMI problem (20). At this point, a robust compensator, which guarantees the robust stability and satisfies the upper bound of the worst case $H_2$ performance, is completely calculated. We then repeat the procedure until the decrease in the upper bound of the worst-case $H_2$ performance is sufficiently small. The solution algorithm to design a set of controllers for systems with real parametric uncertainty satisfying $|\delta_i| < \gamma$ is briefly summarized in figure 2. As discussed in section 3, a bilinear sector transformation (Desoer and Vidyasagar 1975) can be used to convert this problem into a form in which the passivity theorem can be applied.

**Remark 4.1:** The procedure of alternating between the LMI problems is an iterative approach of solving a non-convex optimization problem. It is not guaranteed to converge in general, but in our experience it does converge, although not necessarily to the global optimum. Note that each step of the iteration can be solved very efficiently by a previously developed semidefinite programming algorithm SP (Vandenberghe and Boyd 1994) and very easily coded using a user-friendly interface SDPSOL (Wu and Boyd 1996).

**Remark 4.2:** An important distinction between the V–K iteration and the D–K iteration of the $\mu/K_m$ synthesis is that in our approach there is a shared variable between each iteration: specifically, $\tilde{P}$ is the common variable between the V step and the K step, in which $\tilde{P}$ appears as $P$, $Z$, $Q$, $Y$ and $X$. However, for the D–K iteration the D step (the $\mu/K_m$ analysis) is entirely separate from the K step (the $H_\infty$ synthesis).

## 5. Numerical example

Grocott *et al.* (1994) compared several robust control design techniques using benchmark problems based on a cantilevered Bernoulli Euler beam with unit length

1. Initialize the uncertainty to be zero (a nominal system) and design the controller via Linear Quadratic Gaussian (LQG) or any other robust control design technique.
2. Choose $(A_W, B_W)$ by a method such as one described in §3.3. Initialize $(C_W, D_W)$ by solving (19) where $(A_c, B_c, C_c)$ are fixed.
3. Repeat{ [Outer Loop]
   (a) Repeat{ [Inner Loop]
      i. Solve the optimization problem (27), *i.e.*, solving for $(P, Q, Y, Z, X, P_W)$ where $(C_W, D_W)$ are fixed. Then compute $\tilde{P}$, $B_c$, and $C_c$ using the Completion Lemma.
      ii. Compute $A_c$ by solving a feasibility LMI problem (20).
      iii. Compute $(C_W, D_W)$ by solving (19) where $(A_c, B_c, C_c)$ are fixed.

      } [Inner Loop] Until stopping criterion satisfied.
   (b) Increase the uncertainty to the next desired size and initialize $C_W$ and $D_W$ by the most recent values.

   } [Outer Loop] Until the desired robustness is achieved or the problem is infeasible.

Figure 2. Algorithm of multiplier $H_2$ controller synthesis.

and mass density, and stiffness scaled so that the fundamental frequency is 1 rad/s. The infinite order dynamics of the beam are truncated at four modes, where $\omega_1 = 1$ rad/s, $\omega_2 = 6.27$ rad/s, $\omega_3 = 17.55$ rad/s, $\omega_4 = 34.39$ rad/s, and damping $\zeta = 0.01$. The disturbance input, control input, sensor output and performance output are all collocated at the tip of the beam, and the frequency of the third mode of the system is considered to be uncertain. With ±5% shifts in the modal frequency, there are substantial variations in the system gain and phase in the 17–25 rad/s frequency range (Grocott *et al.* 1994, Banjerdpongchai and How 1996). The uncertainty $\delta$ in this case is a constant real scalar that satisfies $|\delta| < \gamma$, where $\gamma$ is referred to as a guaranteed stability bound. As discussed in section 3, we use a bilinear sector transformation (Desoer and Vidyasagar 1975) to convert this problem into a form in which the passivity theorem can be applied.

The first step in the synthesis process was to design an LQG controller for the nominal system, i.e. $\gamma = 0$, and then apply the technique described in section 3.3 to specify multiplier matrices $A_W$ and $B_W$. Using the procedure in (15), the natural frequency is selected as $\beta_W \approx 17$ rad/s. The damping ratio, $\zeta_W$, is arbitrarily set equal to 0.1. This selection process explicitly shows the plant dependent nature of the stability multiplier. Several controllers were designed using the LMI synthesis approach. Note that for this benchmark problem, each iteration of the outer-loop in the algorithm in section 4.3 required approximately 7 min to execute on a Sun-Sparc 20/60.

There are many interesting aspects to these robust controllers, but we restrict the discussion to a comparison with Popov controllers (Banjerdpongchai and How 1996). Note that both design techniques provide robust stability and performance

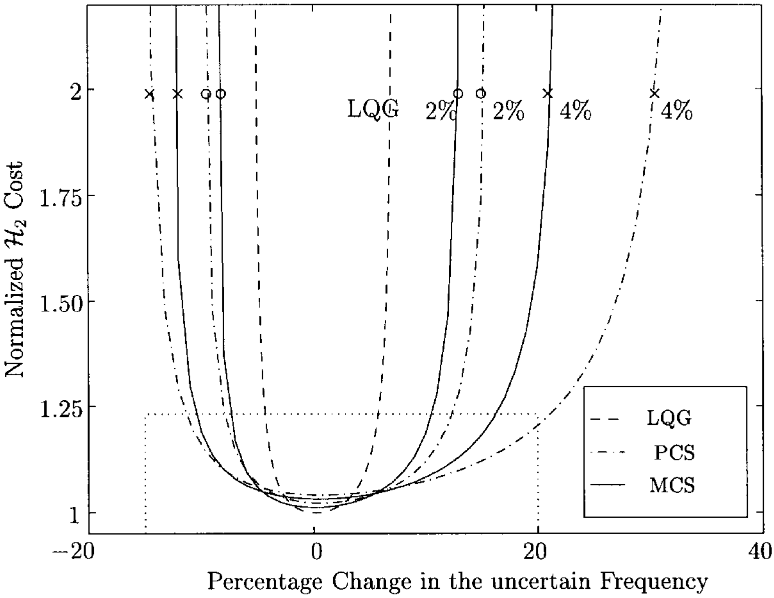


Figure 3. Robust performance plots for LQG, PCS and MCS controllers designed with the symmetric robustness 2% and 4% bounds labelled by the curves.

guarantees for parameter variations within the uncertainty region. However, there is an important distinction between these two techniques. While Popov controllers are designed to capture the memoryless sector bounded nonlinear uncertainty, multiplier controllers directly address real parametric uncertainty (in this example). We will compare the $H_2$ performance of both design techniques for the same guaranteed stability bounds, i.e. comparing the Popov controller with the multiplier controller designed for the equal size of $\gamma$. This consistency is necessary to make a fair comparison between two different design techniques.

There are two distinct quantities that are often used to measure the conservatism of different design techniques. One measure is the increase of the $H_2$ cost in the guaranteed robustness bounds from the nominal $H_2$ cost (i.e. the $H_2$ cost evaluated on the nominal system with the LQG design). The smaller the increase of the $H_2$ cost within the guaranteed region, the less conservative is the control design technique. A second measure of the conservatism is the difference or gap between the guaranteed and achieved stability bounds. The ideal, but not realizable, situation would be to have a non-conservative control design technique that for any size of guaranteed bound yields compensators with a normalized $H_2$ cost, i.e. the $H_2$ cost normalized by the nominal $H_2$ cost, equal to one within the guaranteed region. Although wider achieved stability bounds indicate more robustness of the control designs, the performance achieved outside the guaranteed regions is not directly addressed in the design. Therefore, it could be beneficial to sacrifice the performance outside the guaranteed regions to obtain a better performance inside.

Figure 3 depicts the results obtained for the normalized $H_2$ cost with the given percentage change in the modal frequency. The robust $H_2$ performance of three control design techniques: LQG, Popov and multiplier controller designs are compared. Table 1 summarizes the key points of the robust performance from this plot:

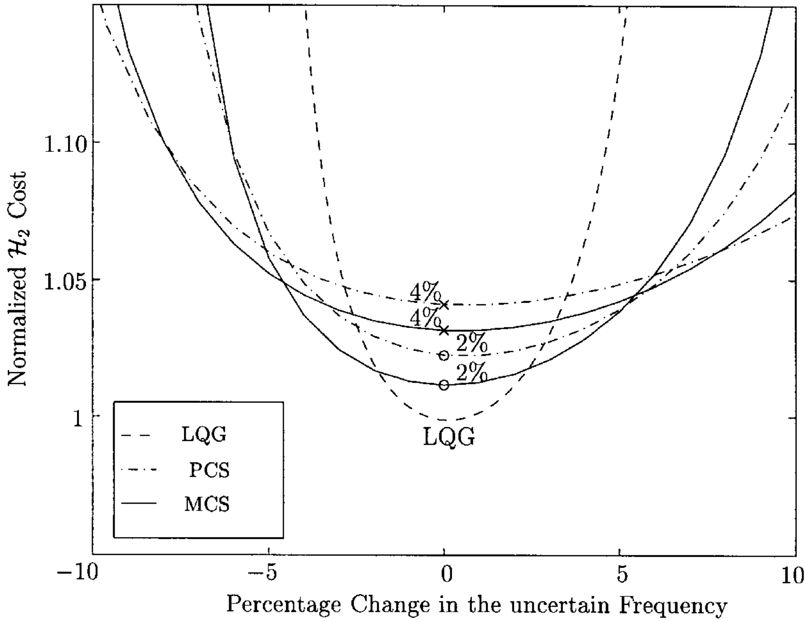


Figure 4. The expanded plot of figure 2 showing the robust performance about the nominal frequency.

| Type of controller | % change of $H_{2,nom}$ cost | Lower stability bound, % | | Upper stability bound, % | |
|---|---|---|---|---|---|
| | | Achieved | Guaranteed | Guaranteed | Achieved |
| LQG | 0 | - 5 | 0 | 0 | 7 |
| PCS2 | 2.36 | - 10 | - 2 | 2 | 16 |
| PCS4 | 4.22 | - 15 | - 4 | 4 | 34 |
| MCS2 | 1.28 | - 9 | - 2 | 2 | 13 |
| MCS4 | 3.28 | - 13 | - 4 | 4 | 23 |

Table 1. Robust stability and performance for the closed-loop system. For consistency, it is necessary to make a fair comparison between two different design techniques for the same guaranteed stability bounds, i.e. comparing the Popov controller with the multiplier controller designed for the equal size of uncertainty.

the percentage change of the $H_2$ cost at the nominal system for Popov and multiplier controller designs compared with the nominal $H_2$ cost, and the lower (upper) achieved and guaranteed stability bounds. From the plot, we note the following two observations.

Comparing the controllers designed using different techniques for the same guaranteed robustness bounds, the normalized $H_2$ cost for the Popov controllers is significantly higher than that for the multiplier controllers in the guaranteed regions. This improvement is clearly shown in the expanded plot in figure 4. As we would like to achieve guaranteed robustness bounds with the minimum possible degradation in the nominal performance, these results indicate that the multiplier controller designs are less conservative than the Popov ones designed for the same guaranteed stability bounds.

For each design, the achieved stability bound is larger than the guaranteed bound but generally these bounds track each other, i.e. a larger guarantee bound is accompanied by a larger stability margin. Figure 3 and table 1 show that, for the same guaranteed bound, the achieved stability bounds for the multiplier controllers are smaller than those achieved by the Popov controllers. Moreover, the difference or gap between the guaranteed and achieved stability bounds of multiplier controllers is smaller than that of Popov controllers. The narrower gap of the multiplier controller designs potentially indicates that the control effort is concentrated on achieving improved performance of the closed-loop system for uncertainty within the guaranteed region. This observation is consistent with the overall design objective.

These two observations strongly support the claim that extending the Popov multipliers to generalized multipliers reduces the conservatism in the robust performance analysis/synthesis for a system with real parametric uncertainty. This is because the generalized multipliers can better capture the uncertainty which is real and constant, whereas the Popov multiplier was devised for a larger class of uncertainty, i.e. sector bounded nonlinearity, which considers real parametric uncertainty as a special case. As a consequence, Popov controller designs yield wider achieved stability bounds than multiplier controller synthesis when real parametric uncertainty is under consideration.

We continue the comparison of the control techniques in terms of the pole and zero location of various robust compensators. This includes the Popov and multiplier controllers at several guaranteed stability bounds. Figure 5 shows that

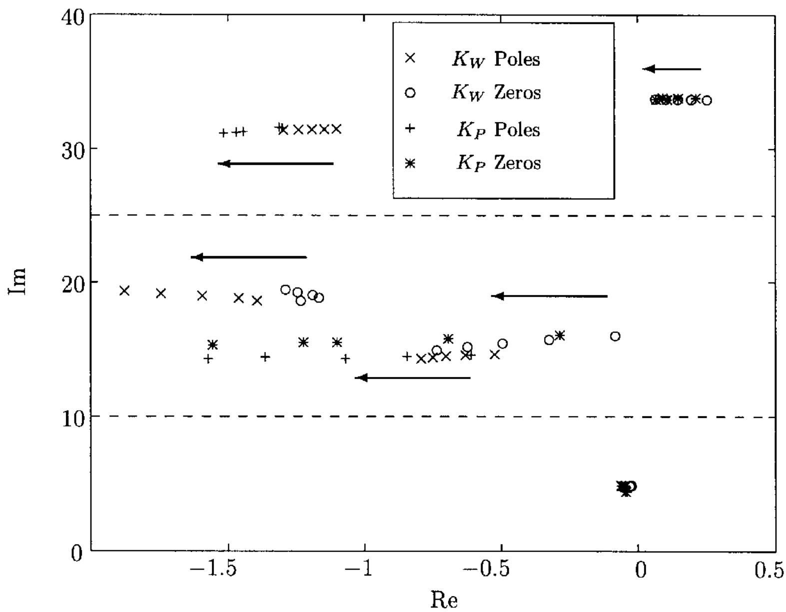


Figure 5. Poles and zeros of the Popov controllers, $K_P$ and multiplier controllers, $K_W$ for the robustness bounds 2, 4, 6, 8 and 10%. The arrows show the direction of change with increasing robustness. Large changes in the uncertain region are clearly evident.

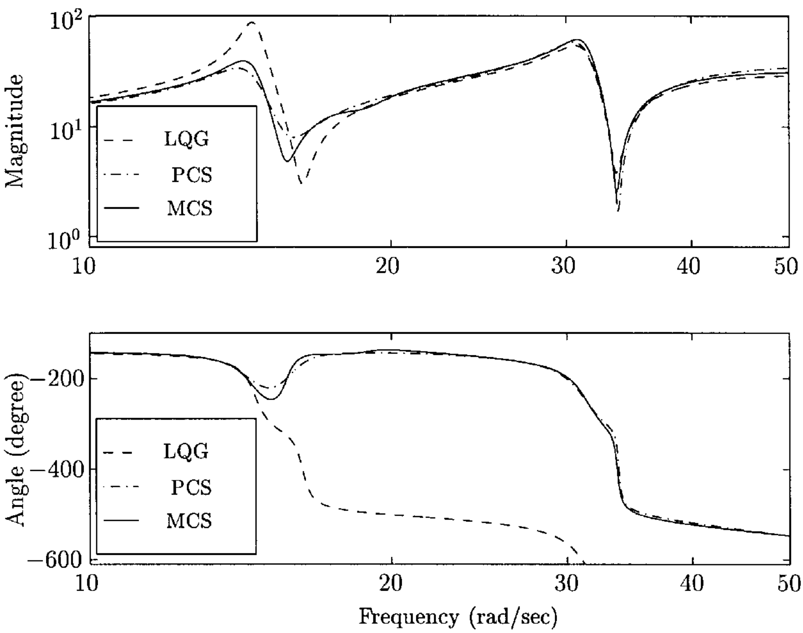


Figure 6. Frequency response of LQG, Popov and multiplier controllers robustified to frequency errors in the third mode. Robust controllers are designed for the uncertainty bound with $\gamma = 0.04$. Significant changes to the response are apparent in the 10–20 rad/s range.

the multiplier controllers have two distinct groups of poles and zeros in the uncertain region (the frequency range local to the uncertain mode). Recall that the full-order multiplier controller has a higher order than the full-order Popov controller by two. The extra compensator poles are at a frequency similar to that of the multiplier dynamics augmented to the system for the analysis test. The figure also shows that the corresponding Popov controllers have only one pole and one zero in this range, and that these controller dynamics become heavily damped as the robustness level is increased. On the other hand, one pole-zero pair of the multiplier controllers is more lightly damped than the Popov controller design, whereas the second pole-zero pair, which was lightly damped initially, becomes more heavily damped as the robustness level is increased. From this plot, we also see that the difference between two control techniques outside the uncertain frequency region is quite small. Figure 6 shows the frequency response of LQG, Popov and multiplier controllers for the uncertainty bound with $\gamma = 0.04$. These graphs show that the differences in the pole-zero patterns in figure 5 lead to subtle changes in the frequency response of the compensators in the uncertain frequency region and at higher frequency. Figure 6 also indicates that the phase of the compensators differs by as much as 25° at approximately 15 rad/s. These plots are interesting because they show how the compensators have been changed, but of course, it is difficult to directly identify how these changes improve the robustness bounds of the closed-loop system.

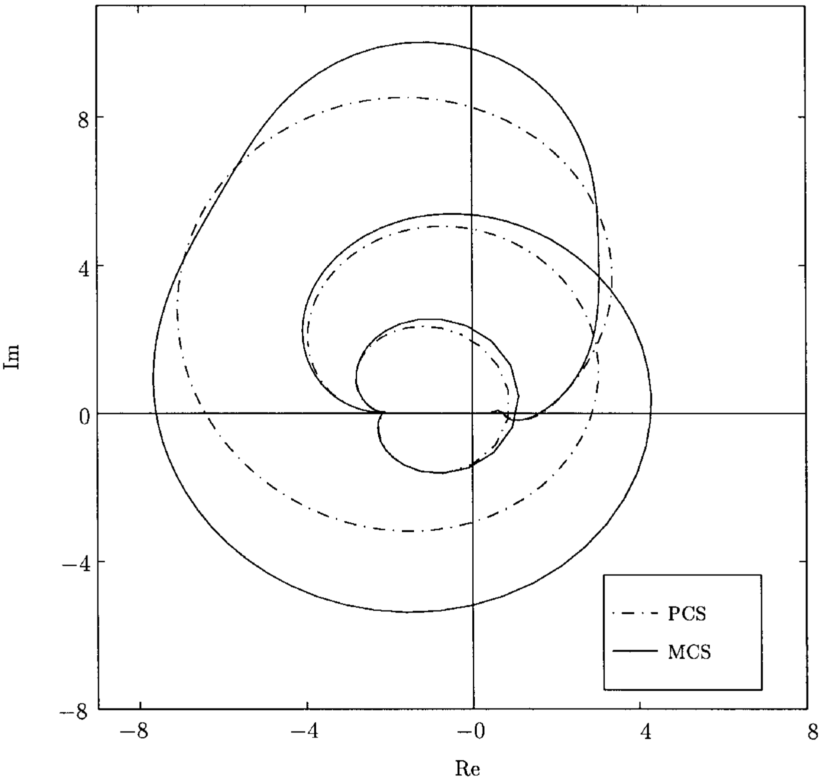


Figure 7. Comparison of the Nyquist plots of the closed-loop transfer function from $p$ to $q$ (across the uncertainty $\Delta$) for Popov and multiplier controller design. Both controllers are designed for the uncertainty bound with $\gamma = 0.04$.

To explore this last point further, a non-conservative real parametric robust analysis must include both magnitude and phase information about the uncertainty $\Delta$. This would be evident in a Nyquist plot. In particular, the inverse of the minimum (maximum) real axis intercept of the Nyquist plot can be used to determine the lower (upper) bound of the real uncertainty that the closed-loop system can tolerate. We show the impact of various compensators on a Nyquist plot of the transfer function from $q$ to $p$ (across the uncertainty $\Delta$) for the closed-loop system. To guarantee the uncertainty bound with $\gamma = 0.04$, the real axis intercepts must lie between $-25$ and 25 on the Nyquist plot. Figure 7 shows that the Nyquist plot of the multiplier controller synthesis almost always encircles the Nyquist plot of the Popov controller synthesis. Furthermore, the real axis intercepts have been increased towards their target of $\pm 25$. Thus, as expected, the controller design with generalized multipliers results in an improvement in the magnitude of the real axis intercepts. This result demonstrates that the control effort of multiplier controllers is exerted more within the guaranteed regions. This observation agrees with the claim that the conservatism of the performance and achieved robustness bounds is reduced when the control synthesis with the generalized multipliers is applied.

## 6. Conclusions

This paper presents an iterative technique for parametric robust $H_2$ control design with generalized multipliers using LMI synthesis. These multipliers better capture information of the uncertainty $\Delta$, which helps to reduce the conservatism in the associated robust performance analysis test for systems with real parametric uncertainty. This approach is limited because not all the multiplier parameters can be optimized during the current synthesis algorithm, but a systematic procedure is provided for selecting the dynamics of multipliers using knowledge of the uncertain systems. This multiplier selection algorithm is a significant first step towards a new, combined analysis and synthesis methodology which extends the prior work on the robust stability and performance analysis. The design procedure is shown to be an effective robust control design technique. In particular, we demonstrate that this new approach produces less conservative compensators than previous Popov controller techniques for a Bernoulli Euler beam with an uncertain modal frequency. A significant advantage of LMI synthesis over the previous procedure using gradient-based optimization techniques is the low overhead associated with developing the optimization conditions. This advantage greatly simplifies the numerical implementation for problems involving the simultaneous optimization of multiplier and controller parameters.

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