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A KYP lemma and invariance principle for systems with multiple hysteresis non-linearities

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Absolute stability criteria for systems with multiple hysteresis non-linearities are given in this paper. It is shown that the stability guarantee is achieved with a simple two part test on the linear subsystem. If the linear subsystem satisfies a particular linear matrix inequality and a simple residue condition, then, as is proven, the non-linear system will be asymptotically stable. The main stability theorem is developed using a combination of passivity, Lyapunov and Popov stability theories to show that the state describing the linear system dynamics must converge to an equilibrium position of the non-linear closed loop system. The invariant sets that contain all such possible equilibrium points are described in detail for several common types of hystereses. The class of non-linearities covered by the analysis is very general and includes multiple slope-restricted memoryless non-linearities as a special case. Simple numerical examples are used to demonstrate the effectiveness of the new analysis in comparison to other recent results, and graphically illustrate state asymptotic stability.

1. Introduction

The Popov stability criteria (Popov 1961) has long been the standard analytical tool for systems having memoryless, sector bounded non-linearities. Details of Popov's analytical approach can be found in the standard texts by Desoer and Vidyasagar (1975), Vidyasagar (1993) and Khalil (1996). When non-linearities, in addition to being sector bounded, are also monotonic and slope restricted, Zames and Falb (1968) proved that the Popov analysis can be further sharpened by employing a more general type of multiplier, often called the Zames–Falb multiplier. Subsequently, Cho and Narendra (1968) found that the existence of such multipliers could be established with an off-axis circle test in the Nyquist plane. While this early work was limited to a scalar non-linearity, an extension by Safonov (1984) considered multiple non-linearities and established criteria through loop shifting and diagonal frequency dependent matrix multipliers, as is now common in the μ/K_m-analysis approach, introduced by Doyle (1982) and Safonov (1982). An alternate approach for the slope restricted case pursued by Singh (1984) and Rasvan (1988) utilized a multiplier first introduced by Yakubovich (1965) for systems with differentiable non-linearities. Although not as general as the Zames–Falb multiplier, the simple form of the Yakubovich multiplier makes it a valuable complement to the Popov analysis. More recently, Haddad and Kapila (1995) and Park *et al.* (1998) have attempted to generalize the results in Singh (1984) and Rasvan (1988) to the case of multiple slope restricted non-linearities. The resulting criteria offered, however, restrict the value of the linear system transfer matrix, $G(s)$, in a variety of ways. In both papers, for instance, the systems are restricted to be strictly proper (i.e. the feedthrough term $D = 0$). Also, in Haddad and Kapila (1995), the value of the system matrix at $s = 0$, $G(0)$ must be either non-singular or identically zero, while in Park *et al.* (1998) the stability guarantee requires that $G(0) = G(0)^{\mathrm{T}} > 0$. In this paper we generalize the analysis for multiple non-linearities in several ways. First we provide the extension to non-strictly proper systems $D \neq 0$ and relax the positivity requirement to $G(0) = G(0)^{\mathrm{T}} > -M^{-1}$, where $M > 0$ is the diagonal matrix of the maximum slopes occurring in the vector of non-linearities. More importantly, we show that the same analysis that applies to the slope restricted case is valid for a class of multiple hysteresis non-linearities as well. This is a rather significant generalization since hysteresis is not sector bounded and has memory, and thus is functionally very different from a memoryless, slope restricted non-linearity. With this result we, in effect, generalize the early scalar hysteresis analysis by Yakubovich (1967) and Barabanov and Yakubovich (1979) and more recent LMI analysis by the authors (Paré and How 1998 a, b), to the case of multiple hysteresis non-linearities.

Using an approach similar to Park *et al.* (1998), we present a linear matrix inequality which, if feasible in a set of free matrix variables, will prove the asymptotic stability of the system. For the slope restricted non-linearity, asymptotic stability means the state converges to the origin, which is assumed to be the unique equilibrium point of the non-linear system. Since a typical hysteresis is in general multivalued, convergence is not to a single point, but rather to a *stationary set*, defined by the intersection of the non-linearity and the dc value of the system matrix. We define these sets explicitly for

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some commonly occurring types of hystereses. In contrast to the previous work of Haddad and Kapila (1995) and Park *et al.* (1998), our Lyapunov function will be a function of the system state, and not its time derivative. This difference results in a more straightforward conclusion of asymptotic stability.

1.1. *Approach overview*

The original general form of Popov's stability criterion (Popov 1961) requires the linear portion of the system to be stable and strictly proper. However, the general form does allow for a single pure integrator in the system. This is sometimes referred to as the *indirect form* or the *indirect control form* of Popov's criteria (see texts by Aizerman and Gantmacher 1964, Narendra and Taylor 1973, Vidyasagar, 1993 for scalar versions), and it commonly has associated with it a three term Lyapunov function. In this paper we will extend this form to the vector case using, as a guide, the procedure of Narendra and Taylor (1973; p. 100) for the single non-linearity, which we summarize in three simple steps. First, we apply a loop transformation that changes the slope sector bounds, differentiates the output of the non-linearity, and results in an integrator state in the transformed linear subsystem, $\tilde{G}(s)$. Provided the original linear subsystem $G(s)$ is stable, $G(s)$ is then cast in Popov's indirect control form. Secondly, we form a three part Lyapunov functional, $V(t)$ that is quadratic in the state of $\tilde{G}(s)$ and includes a particular integral of the non-linearity. When the non-linearity is a hysteresis, having memory, the value of the integral is *path dependent*; while in the memoryless case, it is not. Lastly, the requirement that $\dot{V} \le 0$ is enforced by the existence of a certain LMI, and subsequently, this condition is used to conclude asymptotic stability of certain stationary sets.

The outline of the paper is as follows. First, we characterize the class of non-linearities in the next section, and in particular, limit the hysteresis class to multi-valued functions having an input–output relationship with characteristic loops that circulate in a strict direction. Following that, in §3, the non-linear system is defined and the loop transform used for the analysis is given. The stationary, or equilibrium sets, for the various non-linear systems are in general polytopic regions of state space, and are detailed in §4. This leads directly to the main stability theorem, which is proved in §5. Frequency domain and passivity interpretations of the Lyapunov result are discussed in §6. Simple numerical examples are then presented in §7 which confirm the benefits of our approach with respect to prior stability criteria and give a graphical illustration of the asymptotic stability to the stationary sets.

2. Non-linearities and sector transformations

2.1. *Memoryless, slope restricted*

Following the definition given by Haddad and Kapila (1995), we define the class of non-linearities as

$$\Phi = \left\{ \phi\colon R^m \to R^M \;\middle|\; \begin{array}{l} \phi(y) = [\phi_1(y)_1, \ldots, \phi_m(y_m)]^{\mathrm{T}} \\ \phi \text{ is differentiable a.e. } \in R^m \\ 0 \le \phi_i' < \mu_i,\ i = 1, \ldots, m \\ \phi(0) = 0 \end{array} \right\} \tag{1}$$

The set Φ consists of m decoupled scalar non-linearities, with each scalar component locally slope sector bounded obeying the slop restriction

$$0 \le \frac{\phi_i(y_i^a) - \phi_i(y_i^b)}{y_i^a - y_i^b} \le \mu_i \tag{2}$$

for any $y_i^a,\ y_i^b \in R$. This sector property is sometimes denoted as $\phi_i' \in \text{sector}[0, \mu_i)$, or given the discrete representation (Narendra and Taylor 1973)

$$\Delta\phi_i(y_i)/\Delta y_i \in \text{sector}[0, \mu_i) \tag{3}$$

The slope restriction (3) on a function is a stronger than the standard sector bound condition on a function. This idea is formalized with the following proposition.

Proposition 1 (sector bound property)**:** *A function ϕ_i: $R \to R$ satisfying the conditions $\phi(0) = 0$ and (3) is necessarily sector bounded, with the same bounds. That is, $\phi_i \in \text{sector}[0, \mu_i)$.*

Proof: Simply set $y_i^b = 0$ in (2) and multiply through by $(y_i^a)^2$ to get the relation

$$0 \le \phi_i(y_i^a) y_i^a < \mu_i (y_i^a)^2$$

and thus $\phi_i \in \text{sector}[0, \mu_i)$, which is the standard sector bound condition on ϕ_i. □

Using the approach of Narendra and Taylor (1973) and Zames and Falb (1968), we note that a non-linearity with local slope confined to a finite sector can be converted to a non-linearity with infinite sector width. The transformation requires a positive feedback around the non-linearity, as depicted in figure 1.

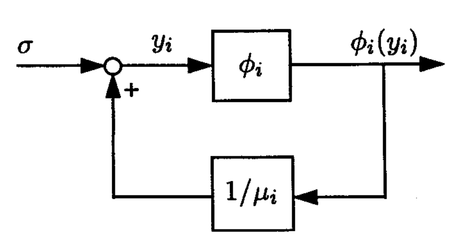


Figure 1. Sector transformation $\tilde{\Phi} \in \text{sector}[0, \infty)$.

Lemma 1 (finite/infinite sector transform)**:** *A slope restricted function* $\phi_i: R \to R$ *with* $\Delta\phi_i(y_i)/\Delta y_i \in \text{sector}[0, \mu_i)$ *under positive feedback with gain* $1/\mu_i$, *as depicted in figure 1, is converted to a non-linearity* $\tilde{\phi}_i: R \to R$ *with the infinite slope bounds satisfying* $\Delta\tilde{\phi}_i(\sigma)/\Delta\sigma \in \text{sector}[0, \infty)$.

Proof: See Narendra and Taylor (1973; pp. 108–109).[1] □

A consequence of Lemma 1 is that the scalar slope functions are non-negative

$$0 \leq \tilde{\phi}_i'(\sigma) < \infty \tag{4}$$

which is equivalent to the sector condition between the time derivatives of the input–output pair

$$0 \leq \dot{\tilde{\phi}}_i \dot{\sigma} < \infty \tag{5}$$

Returning to the vector case, we now apply the same sector transform to each scalar component of ϕ and define a new operator by differentiating the vector output, as depicted in figure 2, where $M = \text{diag}(\mu_1, \ldots, \mu_m) > 0$ is the diagonal matrix of maximum slopes occurring in ϕ. The input–output relation from σ to ξ, as defined in figure 2, is passive, as detailed by the following lemma.

Lemma 2 (passive operator)**:** *Consider a slope restricted non-linearity* $\tilde{\Phi}: R^m \to R^m$ *with decoupled scalar components satisfying* $0 \leq \tilde{\phi}_i'(\sigma) < \infty$. *Then the input–output relation defined with* $\sigma(t)$ *as the input to* $\tilde{\Phi}$ *and output* $\xi(t) = (\mathrm{d}/\mathrm{d}t)\tilde{\Phi}(\sigma)$, *the time derivative of* $\tilde{\Phi}(\sigma)$ (*as depicted in figure 2*) *is passive.*

Proof: For all $T \geq 0$ we have

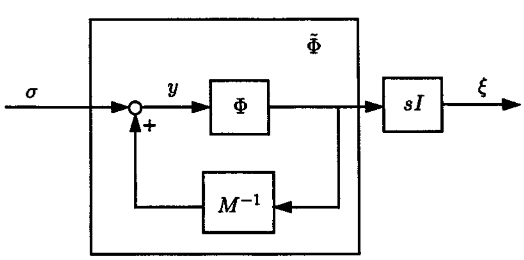


Figure 2. Sector transformation $\tilde{\Phi} \in \text{sector}[0, \infty)$.

$$\int_0^T \sigma^T \xi \, \mathrm{d}t = \sum_{i=0}^{m} \int_0^T \sigma_i \xi_i \, \mathrm{d}t \tag{6a}$$

$$= \sum_{i=0}^{m} \int_0^T \sigma_i \frac{\mathrm{d}}{\mathrm{d}t} \tilde{\phi}_i(\sigma_i) \, \mathrm{d}t \tag{6b}$$

$$= \sum_{i=0}^{m} \int_0^T \sigma_i \tilde{\phi}_i'(\sigma_i) \dot{\sigma}_i \, \mathrm{d}t \tag{6c}$$

$$= \sum_{i=0}^{m} \int_{\sigma_i(0)}^{\sigma_i(T)} \sigma_i \tilde{\phi}_i'(\sigma_i) \, \mathrm{d}\sigma_i(t) \tag{6d}$$

$$= \sum_{i=0}^{m} \left\{ - \int_0^{\sigma_i(0)} \sigma_i \tilde{\phi}_i'(\sigma_i) \, \mathrm{d}\sigma_i(t) + \int_0^{\sigma_i(T)} \sigma_i \tilde{\phi}_i'(\sigma_i) \, \mathrm{d}\sigma_i(t) \right\} \tag{6e}$$

$$\geq -\beta(\sigma(0)) \tag{6f}$$

where

$$\beta(\sigma(0)) = \sum_{i=0}^{m} \int_0^{\sigma_i(0)} \sigma_i \tilde{\phi}_i'(\sigma_i) \, \mathrm{d}\sigma_i(t) \geq 0 \tag{7}$$

since each scalar kernel, $k_i(\sigma_i) = \sigma_i \tilde{\phi}_i'(\sigma_i)$, is a memoryless, sector bounded function, with $k_i \in \text{sector}[0, \infty)$. Therefore, the input–output relation is passive, by the definition given in Desoer and Vidyasagar (1975; p. 73). □

Having now defined the passive transformation for the memoryless class of slope restricted non-linearities, we consider the hysteresis case. In the next section we describe the properties of the hysteresis class and show the very same transformation used for the memoryless case will also convert a vector hysteresis into a passive operator.

2.2. *Hysteresis*

Hysteresis is a property of a wide range of physical systems and devices, such as electro-magnetic fields, mechanical stress–strain elements, and electronic relay circuits. The term *hysteresis* typically refers to the input–output relation between two time-dependent quantities that cannot be expressed as a single-valued function. Instead, the relationship usually takes the form of loops that are traversed either in a *clockwise* or *counter-clockwise* direction. A hysteresis with counter-clockwise loops is sometimes referred to as a *passive hysteresis* (see Hsu and Meyer 1968, p. 366, for example). In general, the output at any given time is a function of the entire past history of the input, and thus unlike the preceding case, hysteresis non-linearities have memory. The memory and loop characteristics of

[1] Note that the sector is half-open, and essentially does not include infinity. More precisely, the transformation should have positive feedback of $1/(\mu - \epsilon)$, where $0 < \epsilon \ll \mu$. This is the approach taken in Zames and Falb (1968), and likewise, we assume this adjustment is included in the sector transform, but for simplicity this will not be expressed explicitly, but is implied by the strict inequality.

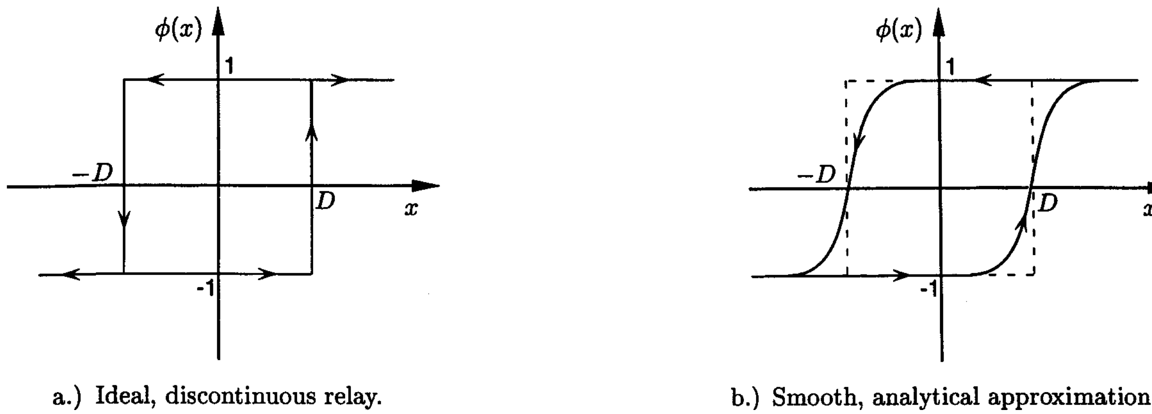


Figure 3. Discontinuous relay replaced with smooth approximation for stability analysis.

hysteresis complicate the analysis to some extent, especially since in practice hysteresis loops can take many forms (Brokate and Sprekels 1996). To simply matters in this section, we assume some additional hysteresis characteristics and thus limit the scope of non-linearities we consider. The class we define, however, still includes many models that occur in practice, such as the hysteretic relay, backlash and Preisach hysteresis (Mayergoyz 1991), which are depicted in figures 3, 4 and 5, respectively. A characteristic common among these non-linearities is counter-clockwise circulation of the input–output relation.[2] In the next section, the assumed characteristics of the scalar non-linearities are detailed, and an example using backlash is given to illustrate the application of the properties. Following that, the vector class of multiple hysteresis non-linearities is defined using the scalar properties.

2.2.1. *Smooth approximation for discontinuities:* Non-linearities with discontinuities, such as the relay depicted in figure 3, can present difficulties for the stability analysis because the transform used (shown in figure 2) involves the time derivative of the non-linear output. As such, the transformed non-linearity will result in an unbounded operator, mapping continuous input signals, with bounded velocities, to an output signal with infinite rate of change. Naturally, this would violate the sector bound (5) established for the memoryless case. In order to use the same analytical approach for hysteresis with discontinuities, certain smooth approximations must be assumed. Smooth approximations for relay-type non-linearities, as depicted in figure 3(*b*), will be assumed for the subsequent analysis, so that the local slope, $\mathrm{d}\phi/\mathrm{d}x = \phi'(x)$, satisfies the bound

$$0 \le \phi'(x) < \infty \tag{8}$$

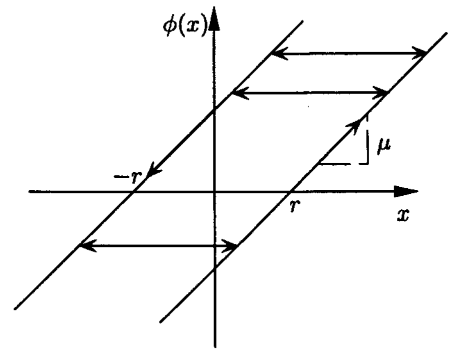


Figure 4. Backlash: deadzone width $= 2r$.

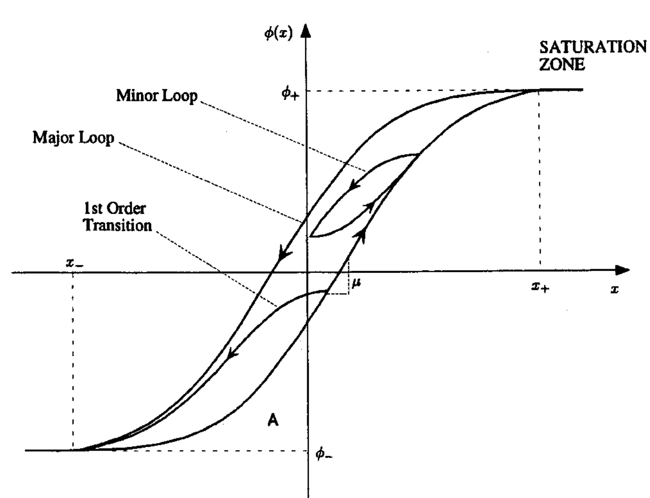


Figure 5. Typical Preisach hysteresis characteristic.

[2] While counter-clockwise circulation is an assumed property of class, it is possible to include clockwise behaviour by employing a coordinate transformation that effectively reverses the circulation, as discussed in Hsu and Meyer (1968; p. 366).

where the upper bound is strict.[3] An important consequence of this condition is that the approximation maps continuous input signals into continuous outputs. This allows us to establish a local Lipschitz condition for the non-linearity. That is, if the input $x\colon \mathbf{R} \to C_0[0,t]$, where $x(t_2)$ is a sufficiently small (local) perturbation of $x(t_1)$ on an interval

$$|x(t_1) - x(t)| < \delta, \qquad \forall t \in [t_1, t_2] \tag{9}$$

then the local Lipschitz condition

$$|\phi(x(t_1)) - \phi(x(t_2))| \le \bar{\phi}'|x(t_1) - x(t_2)| \tag{10}$$

will hold, where $0 \le \phi'(x) \le \bar{\phi}' < \infty$. In this case $\bar{\phi}' = \mu$, the maximum slope appearing in the non-linearity. This property will ensure that the transformed operator is a bounded operator on the space of continuous signals, $C_0[0,t]$,[4] and is required later in § 5 so that the stability proof involves a differentiable Lyapunov function.

2.2.2. *Properties of hysteresis*

Property 1: *Non-local memory.* Unlike memoryless non-linearities, hysteresis output at any given time is a function of the entire history of the input, and the initial condition of the output, ϕ_0. So we define the output $w(t)$ as

$$w(t) = \phi(\phi_0, x([0,t))) \tag{11}$$

$$= \Phi[x, \phi_0](t) \tag{12}$$

At time we will drop the dependence on ϕ_0 for simplicity.

Property 2: *Causality, time invariance and rate independence.* The hystereses considered are causal and time-invariant operators, as given by the standard definitions (Desoer and Vidyasagar 1975). They are also rate-independent, which essentially means that the input–output relation, as depicted on a graph such as figure 5, is unchanged for an arbitrary time scaling of the input function. For instance, the input–output relation described by the relation (x, y) is invariant for changes of the input rate, such as changes in the frequency of cycling. This assumption precludes rate-dependent hysteresis such as the Chua–Stromsmoe model, considered by Safonov and Karimlou (1983), for which the local slope varies with the frequency of the input signal.

Property 3: *Counterclockwise circulation.* Closed loops that occur on the input–output characteristic are strictly counterclockwise. That is, a periodic input $x(t)$, with period $T > 0$, will result in a closed curve relation

$$\int_0^T x(s)\Phi[x]'(s)\,\mathrm{d}s = \int_t^{t+T} x(s)w'(s)\,\mathrm{d}s = \oint_{w(t)}^{w(t+T)} x(s)\,\mathrm{d}w(s) \ge 0 \tag{13}$$

with equality achieved, for the backlash example, when $x(t)$ remains in the backlash deadzone. The value of the integral (13), when the path is closed, is equal to the area enclosed by the hysteresis loop. For partial, unclosed loops, the integral represents the area between the path traversed and the hysteresis output axis (cf. ϕ-axis in figures 3–5).

Property 4: *Positive path integral.* Let Ψ be the intersection of the output ϕ-axis and the hysteresis characteristic curves.[5] Any input–output path $\mathbf{\Gamma} = \{(x(t), w(t)) | t \in [0,T]\}$, originating in Ψ, the path integral $\int_\Gamma x\,\mathrm{d}w$ is non-negative. That is, if $x(t)$, $t \in [0,T]$ with $x(0) = 0$ generates that path $\mathbf{\Gamma}$, joining points $p \in \Psi$ and some arbitrary b, we have

$$\int_0^T x(s)\Phi[x]'(s)\,\mathrm{d}s = \int_0^T x(s)w'(s)\,\mathrm{d}s = \int_{\Gamma_{p\to b}} x\,\mathrm{d}w \ge 0 \tag{14}$$

Similarly, now let $\mathbf{\Gamma}$ denote the path joining any two points on the hysteresis graph, and note that this path may involve many complete cycles, as in (13) above. Let $\mathbf{\Gamma}_{ab}$ denote the shortest path joining the two points a and b, not containing any complete cycles. Assuming $\mathbf{\Gamma}$ results from input $x(t)$, $t \in [0,T]$ and taking a third point $p \in \Psi$, we have that

$$\int_0^T x\Phi[x]'(t)\,\mathrm{d}t = \int^T x(t)w'(t)\,\mathrm{d}t \tag{15a}$$

$$= \int_\Gamma x(t)\,\mathrm{d}w(t) \tag{15b}$$

$$\ge \int_{\Gamma_{ab}} x(t)\,\mathrm{d}w(t) \tag{15c}$$

$$= -\int_{\Gamma_{p\to a}} x(t)\mathrm{d}w(t) + \int_{\Gamma_{p\to b}} x(t)\,\mathrm{d}w(t) \tag{15d}$$

$$\ge -\beta(x(0), \phi_0) \tag{15e}$$

where

[3] See Visintin (1988) for a similar approximation for the hysteretic relay.

[4] See Brokate and Sprekels (1996, p. 24), for a similar discussion.

[5] For the unit relay, figure 3, this set consists of two points: $\Psi = \{(0,1),(0,-1)\}$, for the backlash and Preisach models, Ψ is the corresponding line segment on the ϕ-axis.

$$\beta(x(0), \phi_0) = \int_{\Gamma_{p\to a}} x(t)\,\mathrm{d}w(t) \geq 0 \tag{16}$$

The first inequality (15 *c*) holds from the circulation condition (13), while the second inequality (15 *e*), and the positivity of β is a result of (14).

Property 5: Finally, we require the property that the above Properties 3 and 4 hold when the non-linearity is sector transformed in accordance with Lemma 1. In essence it is required that, under this transformation, the new hysteresis maintains the circulation and positivity properties, but has (half-open) infinite slope sector bound: $0 \leq \tilde{\phi}'(\sigma) < \infty$.

Remarks: The constant β in (16) has the interpretation of the maximum energy that can be extracted (available energy) from the non-linear operator with a given set of initial conditions (Willems 1972). While the properties we assume may appear overly restrictive, many common hystereses have these properties. It can be seen by inspection that the simple relay has Properties 1–4, and, in a trivial manner, it satisfies Property 5 since it is unaffected by the sector transformation. Under the transformation, the Preisach model is re-shaped, with the saturation region maintained and the region of maximum slope μ (see figure 5) becoming vertical; but the circulation and positivity properties still hold. The backlash is a simple analytical model useful to demonstrate these properties, as shown in the following section.

2.2.3. *Energy storage and dissipation for the backlash hysteresis:* Here we show the common backlash nonlinearity conforms to the Properties 1–5 given in the previous section. In particular, we give a simple mathematical representation for the non-linearity, and then show that the positivity constraint (15) holds under the sector transformation indicated by Property 5.

The input–output behaviour of a backlash (figure 4) can be described by two modes of operation, as either tracking or in the deadzone, for which we define

$$\left.\begin{array}{ll} \text{Tracking:} \quad \dot{w} = \mu\dot{y} & \begin{cases} \dot{y} > 0, & w = \mu(y - r) \quad \text{or} \\ \dot{y} < 0, & w = \mu(y + r) \end{cases} \\ \text{Deadzone:} \quad \dot{w} = 0 & |w - \mu y| \leq \mu r \end{array}\right\} \tag{17}$$

where $2r$ is the deadzone width and μ is the slope of the tracking region, as indicated in figure 4. Applying the sector transformation, shown in figure 1, we have, when tracking with positive velocity

$$\sigma\dot{\phi} = \sigma\dot{w} = \left(y - \frac{1}{\mu}w\right)\dot{w} = \mu r\dot{y} \tag{18}$$

and, similarly for negative tracking: $\sigma\dot{w} = -\mu r\dot{y}$. This quantity is then expressed for all times as

$$\sigma\dot{w} = \begin{cases} \mu r|\dot{y}| & \text{when tracking} \\ 0 & \text{in deadzone} \end{cases} \tag{19}$$

Defining the interval $\mathcal{I} = [0, T]$, for some $T \geq 0$, and $T_{trk} \subseteq \mathcal{I}$ encompassing all the subintervals in $\mathcal{I}$ for which tracking occurs, the integral (15) for the backlash becomes

$$\int_0^T \sigma(t)w'(t)\,\mathrm{d}t = \mu r \int_{t\in T_{trk}} |\dot{y}(t)|\,\mathrm{d}t \geq 0 \tag{20}$$

Thus, $\beta = 0$, which means that the sector transformed non-linearity has zero stored (or available) energy. In this case, it can be shown that the transformation induces a dissipation *equality*. In particular, the energy balance, as noted by Brokate and Sprekels (1996, p. 69), is given as

$$\mathcal{M}'(t) - \mathcal{U}'(t) = |\mathcal{D}'(t)| \tag{21}$$

where the terms from (18)–(19) are identified with: $\mathcal{M}'(t) = \dot{w}y$ as the mechanical work rate; $\mathcal{U}'(t) = (1/\mu)w\dot{w}$, the rate of *hysteresis potential* energy storage; and $\mathcal{D}'(t) = \mu r\dot{y}$ as the energy dissipation into the hysteretic element. The transformation (figure 2) strips the energy potential and leaves only the energy dissipation term in the integrand in (20). Expressed this way, we can exactly account for all energy components associated with the non-linear operator. Explicit potential, work and dissipation expressions for more complicated hysteresis operators, such as the Preisach and Prandtl models, is discussed in Brokate and Sprekels (1996). While being very powerful analytical tools, they are not pursued further herein.

2.2.4. *Multiple hysteresis non-linearities:* Having defined all the properties of the scalar hysteresis nonlinearities, defining the class for the vector case is straightforward. We define Φ_h, the multiple hysteresis class as

$$\Phi_h = \left\{ \phi : R^m \to R^m \left| \begin{array}{l} \phi(y) = [\phi_1(y_1), \ldots, \phi_m(y_m)]^{\mathrm{T}} \\ \phi_i \text{ is differentiable a.e. in } R \\ 0 \leq \phi_i' < \mu_i, \quad i = 1, \ldots, m \\ \phi_i \text{ has Properties 1–5} \end{array} \right. \right\} \tag{22}$$

The set Φ_h consists of m decoupled scalar non-linearities, with each scalar component locally slope bounded (wherever the non-linearity is differentiable) and conforming to the properties detailed in the previous section.

Lemma 3 (passive operator, hysteresis case): *Consider a vector hysteresis non-linearity $\Phi_h: R^m \rightarrow R^m$ in the class defined* (22). *Then the input–output relation of the sector transformed operator $\tilde{\Phi}_h$ defined with $\sigma(t)$ as the input to $\tilde{\Phi}_h$ and output $\xi(t) = (\mathrm{d}/\mathrm{d}t)\tilde{\Phi}_h(\sigma)$, the time derivative of $\tilde{\Phi}_h(\sigma)$ (as depicted in figure 2) is passive.*

Proof: For all $T \geq 0$

$$\int_0^T \sigma^{\mathrm{T}} \xi \, \mathrm{d}t = \sum_{i=0}^{m} \int_0^T \sigma_i \frac{\mathrm{d}}{\mathrm{d}t} \tilde{\phi}(\sigma_i) \, \mathrm{d}t \tag{23a}$$

$$= \sum_{i=0}^{m} \int_0^T \sigma_i w_i'(t) \, \mathrm{d}t \tag{23b}$$

$$= \sum_{i=0}^{m} \int_{\Gamma_i} \sigma_i(t) \, \mathrm{d}w_i(t) \tag{23c}$$

$$\geq \sum_{i=0}^{m} \int_{\Gamma_{ab_i}} \sigma_i(t) \, \mathrm{d}w_i(t) \tag{23d}$$

$$= \sum_{i=0}^{m} \left\{ - \int_{\Gamma_{p_i \rightarrow a_i}} \sigma_i(t) \, \mathrm{d}w_i(t) + \int_{\Gamma_{p_i \rightarrow b_i}} \sigma_i(t) \, \mathrm{d}w_i(t) \right\} \tag{23e}$$

$$\geq -\beta(\sigma(0)) \tag{23f}$$

where

$$\beta(\sigma(0), w(0)) = \beta(y(0), \phi_0) = \sum_{i=0}^{m} \int_{\Gamma_{p_i \rightarrow a_i}} \sigma_i(t) \, \mathrm{d}w_i(t) \geq 0 \tag{24}$$

according to Properties 4 and 5 of the class. Hence, the input–output relation is passive, by the definition given in Desoer and Vidyasagar (1975, p. 173). □

Note that the proof is structured in a way analogous to the memoryless case. Instead of positive (sector bounded, path independent) line integrals, the corresponding steps here involve positive path integrals.

3. System description and loop transformation

As in the standard absolute stability analysis framework, it is assumed that the non-linearity can be isolated from the linear dynamics and placed into a feedback path, as is shown in figure 6(*a*). Assuming the linear dynamics $G(s)$ has a minimal state space representation (A, B, C, D), with A Hurwitz, the non-linear (Lur'e) system is described as

$$\left. \begin{aligned} \dot{x} &= Ax + Be \\ y &= Cx + De \\ p_i(t) &= \phi_i(y_i(t)), \qquad i = 1, \dots, m \end{aligned} \right\} \tag{25}$$

where $p(t) \in R^m$ and $\phi \in \Phi$, as defined by either the multiple memoryless or hysteresis class, as before. In order to convert the non-linearity into a passive operator, in accordance with Lemmas 2 and 3, we introduce the loop transform, as described in figure 2, to give the equivalent

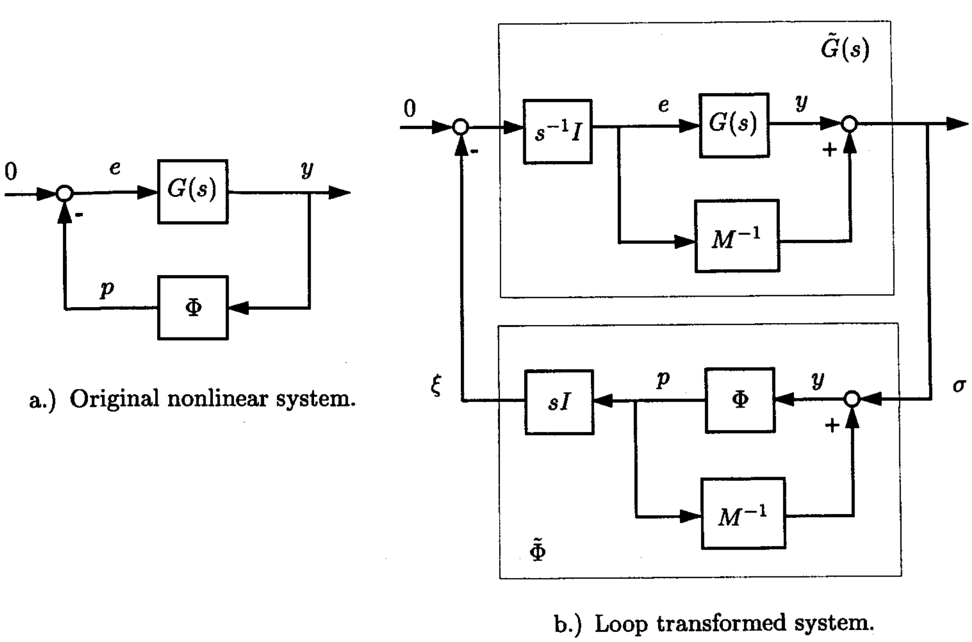


Figure 6. Non-linear system and loop transformation.

system shown in figure 6(*b*). Note that $\tilde{\Phi}$ is now passive, and that the transformed linear system

$$\tilde{G}(s) = (G(s) + M^{-1})(s^{-1}I) \tag{26}$$

has the state space representation

$$\tilde{G} \stackrel{s}{=} \left[\begin{array}{c|c} \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} & \begin{bmatrix} B \\ D + M^{-1} \end{bmatrix} \\ \hline [0 \quad I] & 0 \end{array}\right] \tag{27}$$

By the Hurwitz assumption, we have that A is invertible, and thus by introducing the similarity transform

$$T = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix}$$

the augmented system $\tilde{G}(s)$ can be decomposed into its stable and constant dynamic components as

$$\tilde{G}(s) = \tilde{G}_r(s) + s^{-1}R \tag{28}$$

where $R = G(0) + M^{-1}$ with $G(0) = -CA^{-1}B + D$, and the stable component G_r is reduced by the integrator states and has the state space description

$$\tilde{G}_r \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline CA^{-1} & 0 \end{array}\right] \tag{28}$$

With the linear dynamics decomposed in this way, the non-linear, closed loop system can then be expressed in the vector version of Popov's indirect control form (see Vidyasagar 1993, p. 231, for example), as is depicted in figure 7. The dynamics of the original Lur'e system (25) corresponding now to the Popov form are equivalently given as

$$\left.\begin{aligned} \dot{x} &= Ax + Bu = Ax - B\xi \\ \dot{\eta} &= -\xi, \qquad \eta(0) = -\phi_M(0) \\ \sigma &= CA^{-1}x + R\eta \\ \xi &= \dot{\phi}_M(t) \end{aligned}\right\} \tag{30}$$

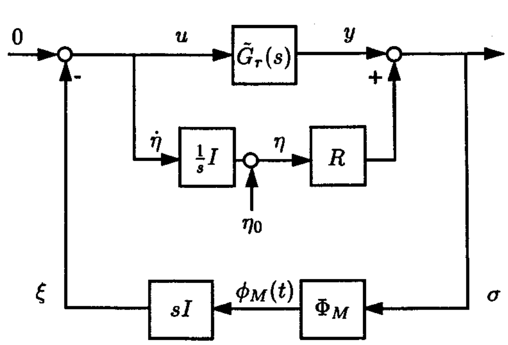


Figure 7. Popov indirect control form.

Proper initialization of the integral state η, as shown in figure 7, leads to the identities

$$\eta(t) = -\phi_M(t) \tag{31 a}$$

$$\dot{\eta}(t) = -\xi = -\dot{\phi}_M(t) \tag{31 b}$$

The stable (equilibrium) conditions for the hysteresis case differs from the memoryless, slope-restricted because the hysteresis is multivalued. As a result, while the equilibrium point for the memoryless non-linear system is unique, convergence for the hysteresis system is to an invariant set, which may consist of an infinite number of points. The next section provides explicit descriptions of these stability sets.

4. Stationary sets and stability definitions

Stability theory is often used to determine whether or not an autonomous system will achieve some sort of steady state condition. Generally speaking, in steady state, the system state may be at an equilibrium point (at rest with $\dot{x} = 0$), or in a limit cycle. In either case, the state $x(t)$ belongs to an invariant set (Hahn 1963, Vidyasagar 1993). The *largest* invariant set $\mathsf{M} \subset R^n$, for a particular system, is the union of all equilibrium points and the sets containing all possible limit cycles. The equilibrium, or stationary, set $\mathsf{E} \subseteq \mathsf{M}$, for the non-linear system (25) is defined as

$$\mathsf{E} = \{x \in R_n \mid \text{such that (33) is satisfied}\} \tag{32}$$

where (33) is the set of algebraic conditions

$$y_{ss} = [-CA^{-1}B + D]e_{ss} = G(0)e_{ss} \tag{33 a}$$

$$e_{ss} = -\phi(y_{ss}) \tag{33 b}$$

$$x_{ss} = -A^{-1}Be_{ss} \tag{33 c}$$

Naturally, E is unique to each system (25) and, in particular, depends on the type of non-linearity present. Various stationary sets are given below.

4.1. *Stationary set for memoryless non-linearity*

For the slope-restricted non-linearity, we assume there exists a unique equilibrium point $x = 0$, for the closed loop system (25). That is, E_m is a singleton

$$\mathsf{E}_m = \{0\} \tag{34}$$

This result is consistent with the sector bounded property of the class Φ, and the assumption $G(0) > -M^{-1}$. Geometrically, this condition means that the graph of the ith non-linearity $\phi_i(y_i)$ and the line $\phi_i = -y_i/G_{ii}(0)$ intersect only once, at the origin. This intersection is necessarily non-unique in the hysteresis case, and as a result, E_h is comprised of finite regions in state space. These sets are defined below for various special cases.

4.2. *Stationary sets for hysteresis non-linearities*

The stationary sets for multiple hysteresis can be defined with a simple extension of the graphical technique for the scalar case originally detailed by Barabanov and Yakubovich (1976).[6] To proceed, consider a generic Preisach non-linearity, and note that conditions (33 *a–b*) together can be depicted graphically, as shown in figure 8, as the intersection of the line $\phi_i = -y_i/G_{ii}(0)$ and the graph of the hysteresis. This intersection defines the range of outputs for each non-linearity $\phi_i \in [\underline{\phi}_i, \bar{\phi}_i]$ which must be satisfied simultaneously for each ϕ_i, $i = 1, \ldots, m$. Then letting each ϕ_i vary over the allowed range maps out the invariant set E, according to the condition (33 *c*) $x = -A^{-1}Be$, where $e = -\phi$. Note that if $g_{ii}(0) = 0$, then corresponding limits $\underline{\phi}_i$, $\overline{\phi}_i$ are simply the extreme values of intersection of the hysteresis with the ϕ-axis. The stationary sets for the relay, backlash and Preisach hysteresis non-linearities are given next.

4.2.1. *Hysteretic relay:* For a system with a bank of m unit relays, as shown in figure 3, the stationary set is given by

$$\mathsf{E}_{\text{relay}} = \left\{ x \in R^n \,\middle|\, \begin{array}{l} x = -A^{-1}Be \\ e \in R^m, \quad e_i \in \{-1, 1\}, \quad i = 1, \ldots, m \end{array} \right\} \tag{35}$$

$\mathsf{E}_{\text{relay}}$ consists of 2^m discrete points in R^n. Each point is essentially the steady state solution of the open loop system $G(s)$ in response to a particular constant input vector e consisting of elements $e_i = +1$, or -1.

4.2.2. *Backlash and Preisach non-linearities:* The equilibrium sets for these two type of non-linearities are defined in the same way, since both operators admit outputs that range continuously over a prescribed interval. Once the output limits are defined, the stationary set is completely determined.

$$\mathsf{E}_{\text{backlash}}, \mathsf{E}_{\text{Preisach}} = \left\{ x \in R^n \,\middle|\, \begin{array}{l} x = -A^{-1}Be \\ e \in R^m, \quad e_i \in [\underline{\phi}_i, \overline{\phi}_i], \quad i = 1, \ldots, m \end{array} \right\} \tag{36}$$

Note that these sets are polytopic regions, and are equivalently defined as the convex hull of the corresponding set of limiting vectors

$$\mathsf{E}_{\text{backlash}}, \mathsf{E}_{\text{Preisach}} = \mathbf{Co}\{\underline{v}_i \overline{v}_i, \ldots, \underline{v}_m, \overline{v}_m\} \tag{37}$$

where $\underline{v}_i$, $\overline{v}_i \in R^n$, with

$$\underline{v}_i = -AB\underline{z}^{-1}, \qquad \text{where } \underline{z}_j = \begin{cases} \underline{\phi}_i, & j = i \\ 0, & \text{else} \end{cases}$$

and $\overline{v}_i$ defined similarly.

The definitions for the stationary sets E provide a clear idea of the position of $x \in R^n$ should the system achieve the equilibrium condition defined by $\dot{x} = 0$. Before providing the stability criteria that guarantees the system is indeed stable, we give precise definitions of what it means for a system to be stable with respect to an invariant set.

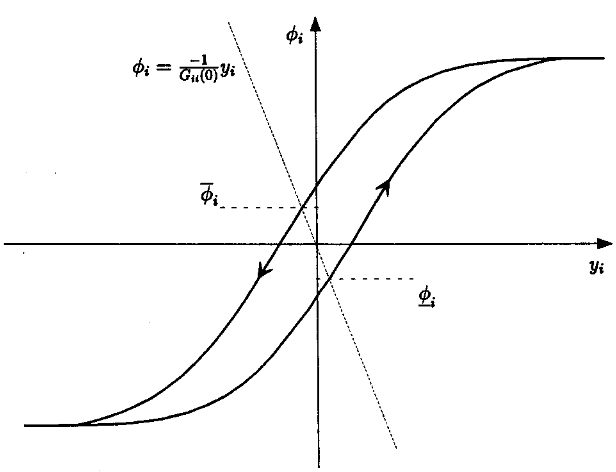


Figure 8. Graphical criteria for determining E.

[6] A similar definition for (32) is given in Jönsson (1998).

4.3. *Definitions of stability*

Using standard notation (as by Hahn 1964, for example), define the trajectory of motion for an initial condition $x(0) = x_0$ of some arbitrary system as $q(x_0, t)$. For an invariant set M of the system, the distance to the set from any arbitrary point is given by

$$\text{dist}(x, \mathsf{M}) = \inf |x - y|, \qquad y \in \mathsf{M}$$

with $\text{dist}(x, \mathsf{M}) = 0$ for $x \in \mathsf{M}$. A closed invariant set M is called *stable*, if for every $\epsilon > 0$ a number $\delta > 0$ can be found such that for all $t > 0$

$$\text{dist}(q(x_0, t), \mathsf{M}) < \epsilon$$

provided

$$\text{dist}(x_0, \mathsf{M}) < \delta$$

If in addition

$$\text{dist}(q(x_0, t), \mathsf{M}) \to 0, \qquad \text{as } t \to 0$$

then M is said to be *asymptotically stable*.

5. Stability theorem

This section provides a Lyapunov-based asymptotic stability theory for the systems with either slope-restricted (memoryless) or hysteresis non-linearities. The Lyapunov function used refers to the transformed system defined in § 3 and includes the integral of the non-linearity that is positive, as a result of the passive properties defined in §2. Negativity of the Lyapunov derivative is enforced by a certain matrix inequality of a form similar to that associated with the well-known KYP Lemma (see Boyd *et al.* 1994, p. 120, for one treatment). The theorem then concludes asymptotic stability of the origin in the case of the memoryless, slope-restricted non-linearities, and for the equilibrium sets given in §4.2 in the hysteresis case by using the Lyapunov conditions and employing basic analytical results.

Theorem 1 (asymptotic stability): *If there exist constants P, N, Δ, with*

$$\left.\begin{array}{ll} P \in R^{n\times n}, & P = P^{\mathrm{T}} > 0 \\ \Delta \in R^{m\times m}, & \Delta = \Delta^{\mathrm{T}} > 0 \\ N = \mathrm{diag}(n_1, \ldots, n_m), & n_i > 0, \quad i = 1, \ldots, m \end{array}\right\} \quad (38)$$

such that

$$\begin{bmatrix} -A^{\mathrm{T}}P - PA & C^{\mathrm{T}}N + A^{-\mathrm{T}}C^{\mathrm{T}} - PB \\ (\cdot)^{\mathrm{T}}_{12} & ND + D^{\mathrm{T}}N + 2NM^{-1} - \Delta \end{bmatrix} \geq 0 \quad (39)$$

and $R = R^{\mathrm{T}} > 0$, $R = G(0) + M^{-1}$, then the closed loop system (30) *is asymptotically stable. In this case, the Lyapunov functional*

$$V(x(t), \eta(t), t) = x(t)^{\mathrm{T}}Px(t) + 2\int_0^t \sigma^{\mathrm{T}}(\tau)\xi(\tau)\,\mathrm{d}\tau + \beta(\sigma_0, \xi_0) + \eta^{\mathrm{T}}(t)R\eta(t) \quad (40)$$

proves stability.

Proof: Choosing β as (7) for the slope-restricted non-linearity, or as (24) when the non-linearity is a multiple hysteresis,[7] then $V \geq 0$; and since $P, R > 0$, $V \to \infty$ whenever $(x, \eta) \to \infty$, so V is positive definite and hence, a valid Lyapunov candidate. In order to assert $\dot{V} \leq 0$, first note that matrix inequality (39) implies, for all $x \in R^n$, $u \in R^m$

$$x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x \leq 2x^{\mathrm{T}}(C^{\mathrm{T}}N + A^{-\mathrm{T}}C^{\mathrm{T}} - PB)u + u^{\mathrm{T}}M_{22}u \quad (41)$$

where M_{22} is the (2, 2) entry of the LMI (39). Using this fact, and (31) we have

$$\begin{aligned} \dot{V}(x, \eta) &= x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x - 2x^{\mathrm{T}}PB\dot{\phi}_M + 2\sigma^{\mathrm{T}}\dot{\phi}_M + 2\eta^{\mathrm{T}}R\dot{\eta} \\ &\leq -2x^{\mathrm{T}}(C^{\mathrm{T}}N + A^{-\mathrm{T}}C^{\mathrm{T}})\dot{\phi}_M + 2\sigma^{\mathrm{T}}\dot{\phi}_M + 2\phi_M^{\mathrm{T}}R\dot{\phi}_M \\ &\quad + \dot{\phi}_M^{\mathrm{T}}M_{22}\dot{\phi}_M \qquad (42a) \\ &= -2(\dot{\sigma} + (D + M^{-1})\dot{\phi}_M)^{\mathrm{T}}N\dot{\phi}_M \\ &\quad - 2(\sigma + R\phi_M)^{\mathrm{T}}\dot{\phi}_M + 2\sigma^{\mathrm{T}}\dot{\phi}_M + 2\phi_M^{\mathrm{T}}R\dot{\phi}_M \\ &\quad + \dot{\phi}_M^{\mathrm{T}}M_{22}\dot{\phi}_M \\ &= -2\dot{\sigma}^{\mathrm{T}}N\dot{\phi}_M - \dot{\phi}_M^{\mathrm{T}}\Delta\dot{\phi}_M \\ &\leq -\dot{\phi}_M^{\mathrm{T}}\Delta\dot{\phi}_M \qquad (42b) \\ &\leq -\delta\,|\dot{\phi}_M|^2 \qquad (42c) \\ &\leq 0 \qquad (42d) \end{aligned}$$

where the first inequality (42 *a*) is due to the LMI condition, the second (42 *b*) a result of the time-derivative sector condition (5), and the last two (42 *c*–*d*) follow from the constraint $\Delta > 0$ (38), and the assumption that δ is the minimum eigenvalue of Δ. Now since V is positive definite in x, η and $\dot{V} \leq 0$, we conclude the closed loop system is stable, or, simply that x and η are bounded. To find asymptotic stability, first note that

$$\dot{V} \leq -\delta|\dot{\phi}_M|^2 \quad \Rightarrow \quad \dot{\phi}_M(t) \to 0 \quad \text{as } t \to \infty \quad (43)$$

since $V(t)$ is bounded below.[8] Further, using (42 *c*), we have

$$V(t) - V(0) \leq -\delta\int_0^t |\dot{\phi}_M|^2\,\mathrm{d}t \quad (44)$$

which, can be rewritten as

$$\int_0^t |\dot{\phi}_M|^2\,\mathrm{d}t \leq \frac{1}{\delta}(V(0) - V(t)) \leq V(0)/\delta \quad (45)$$

which implies $\dot{\phi}_M \in \mathcal{L}_2$, and as a result $y(t) \in \mathcal{L}_2$ as well since $\tilde{G}_r$ is $\mathcal{L}^2$-stable (i.e. A Hurwitz). Using the system dynamics (30), the signal y and its derivative are expressed as

$$y(t) = CA^{-1}\left[\mathrm{e}^{At}x(0) + \int_0^t \mathrm{e}^{A(t-\tau)}\,Bu(\tau)\,\mathrm{d}\tau\right] \quad (46a)$$

$$\dot{y}(t) = C\mathrm{e}^{At}x(0) - C\int_0^t \mathrm{e}^{A(t-\tau)}\,B\dot{\phi}_M(\tau)\,\mathrm{d}\tau - CA^{-1}B\dot{\phi}_M(t) \quad (46b)$$

[7] In the particular case when the non-linearity is of the multiple backlash type, $\beta = 0$, as discussed in §2.2.3

[8] Consider that, if $\dot{\phi}_M$ did not tend to zero, then $V(t)$ would eventually become negative, and violate the constraint $V(t) \geq 0$.

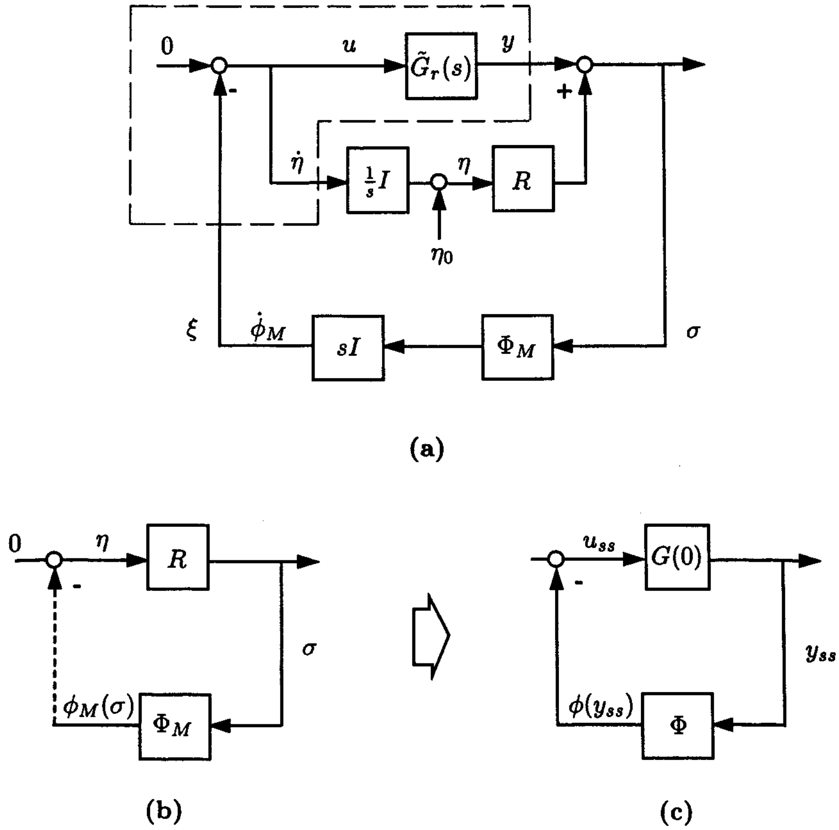


Figure 9. The condition $\dot{V} \equiv 0$ implies steady state condition on the Popov system. The signals contained in the dashed region seen in (a) tend to zero asymptotically. In the limit, this allows reduction to the system (b), which is the transform equivalent to (c), that describes the steady state equilibrium condition (47).

Assuming Lipschitz continuous non-linearities, as discussed in §2.2.1, so that $\dot{\phi}_M(t)$ exists (i.e. $\dot{\phi}_M(t) \in \mathcal{L}_\infty$), we have that $\dot{y} \in \mathcal{L}_\infty$.[9] In this case, the two conditions $y(t) \in \mathcal{L}_2$, $\dot{y} \in \mathcal{L}_\infty$ imply that $y(t) \to 0$ as $t \to \infty$ (see, for example, Narendra and Annaswamy 1989, Lemma 2.1.2). The asymptotic conditions $y(t)$, $\dot{\phi}_M(t) \to 0$ together require that the closed loop system must approach an equilibrium condition as $t \to 0$. To see this, first note that the conditions $\xi = -\dot{\phi}_M \to 0$ and $y \to 0$ imply that all signals of the Popov system (30) contained within the dashed region of the block diagram in figure 9(a) approach zero asymptotically. Secondly, $-\dot{\xi} = \dot{\eta} \to 0$, together with the condition $\dot{\eta}(t) \in \mathcal{L}_1$, established in §9, implies that $\lim_{t\to\infty} \eta(t)$ ex-ists, and is a constant vector. Further, recall that the initialization of variable $\eta(0) = -\phi_M(0)$ implies that $\eta(t) = -\phi_M$ $\forall t \geq 0$, as given by equation (31). Thus, in the limit, the zero signals can be eliminated and the system reduced to that shown in figure 9(b), where the signal equivalence mentioned above is indicated by the dashed line. Reversing the sector transformation further simplifies the diagram to that in figure 9(c), which corresponds to the equivalent algebraic conditions

$$y_{ss} = G(0)u_{ss} \tag{47a}$$

$$u_{ss} = -\phi(y_{ss}) \tag{47b}$$

$$\bar{x} = -A^{-1}Bu_{ss} \tag{47c}$$

which are identical to the conditions (33) that describe the stationary set E. Therefore, in the hysteresis case, we conclude global asymptotic stability of the set E. In the special case of the system with multiple slope-restricted non-linearities, the set E is simply the origin, as noted by equation (34). □

Remarks:

(1) This proof utilizes a combination of Lyapunov and input–output stability theories. Of course, connections between Lyapunov and input–output stability concepts have been well established (Willems 1971 b, Hill and Moylan 1980, Boyd and Yang 1989). In this case, passivity conditions are used to establish Lyapunov stability

[9] Recall continuous approximation to establish local Lipschitz condition (10), and see Remark 2 below for alternate treatment allowing for discontinuities.

arguments for slope restricted/hysteresis non-linear systems, all within the analytical framework of Popov's indirect control form. An alternate approach could proceed using passivity (as is done in Paré and How 1998 *b*) or Popov's hyperstability theorem (Popov 1973), exclusively. However, the Lyapunov component included here enables the additional conclusion of asymptotic stability of the set E. Positive real and passivity interpretations of the analysis are further explored in the following section.

(2) Note that the hysteresis set Φ_h (22) includes the hysteretic relay, which as a discontinuous input–output mapping. Strictly speaking, the proof given does not apply to such non-linearities directly. In order to maintain simplicity, we will assume in these cases that discontinuities can be replaced with a reasonably smooth approximation so that Lipschitz conditions are satisfied (see Visintin 1988 for a similar approximation for the hysteretic relay). A more rigorous approach, could be developed for these discontinuous non-linearities using Filippov (1988) state solutions, one-sided Lyapunov derivatives as described by Hahn (1963) and Clarke (1983), and the generalized version of LaSalles invariance principle (LaSalle 1976).

(3) The condition $R = R^{\mathrm{T}} > 0$ is not overly restrictive. For instance, the off-diagonal elements $G(s)$ can often be arbitrarily scaled using diagonal scaling matrices. In this way the matrix $G(0)$ can be made symmetric with the necessary gain adjustments incorporated into the non-linearity. The condition $R = R^{\mathrm{T}} > 0$ is less restrictive than the condition $G(0) = 0$ given by Haddad and Kapila (1995), and the criterion $G(0) = G(0)^{\mathrm{T}} > 0$ required by Park *et al.* (1998), whenever the non-linearity has finite maximum slope. The criteria in Park *et al.* (1998) includes the additional constraint that $NG(0) = G(0)^{\mathrm{T}}N$, which limits N to a scalar quantity in the case when $G(0)$ is a full matrix. This can further restrict the analysis, as is illustrated with a simple example in § 7.

6. Passivity and frequency domain interpretations

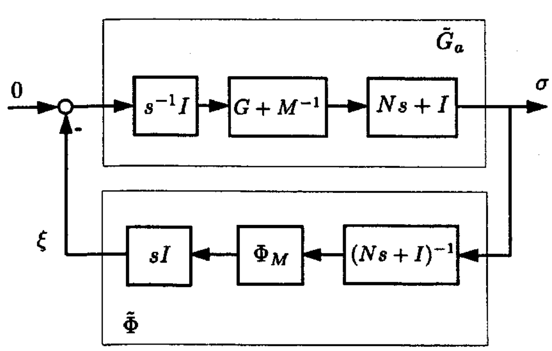
The LMI (39) is recognized as a strict passivity condition on the linear system

$$\tilde{G}_{ra} = \left[\begin{array}{c|c} A & B \\ \hline NC + CA^{-1} & N(D + M^{-1}) \end{array}\right] \tag{48}$$

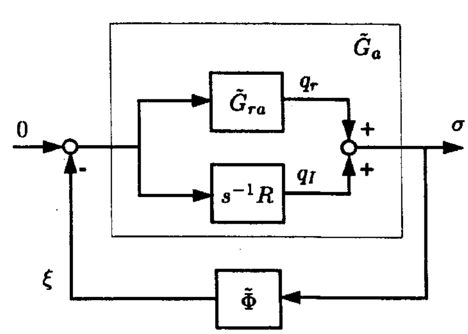
which is an augmented version of the reduced system $\tilde{G}_r$. Strict passivity of this augmented system is a requirement for stability which could have been derived using an equivalent analysis of the system shown in figure 6 by introducing multipliers, following the method detailed in Willems (1971 a; Ch. 6). A robust stability analysis using passivity and multipliers for specific case of systems with a single hysteresis was recently done by the authors (Paré and How 1998 b). To proceed, introduce the multiplier $W(s) = Ns + I$ (with N as defined in (38)) into the transformed system, as shown in figure 10(*a*). In this case, premultiplying the hysteresis Φ_M with W^{-1} as shown results in a new non-linearity $\tilde{\Phi}$ in the feedback path which is passive. This passivity condition is ascertained using the steps in the proofs of Lemmas 2 and 3 and using the additional time derivative constraint (5). Introducing the multiplier similarly leads to the transformed linear system

$$\begin{aligned} \tilde{G}_a(s) &= W(s)(G(s) + M^{-1})(s^{-1}I) \\ &= (Ns + I)(G(s) + M^{-1})(s^{-1}I) \end{aligned} \tag{49}$$

This decomposes, as was done in equations (26)–(28), to



a.) Augmented, transformed nonlinear system.



b.) System in Popov Indirect form.

Figure 10. Augmented, passive system.

$$\tilde{G}_a(s) = \tilde{G}_{ra}(s) + s^{-1}R \qquad (50)$$

where again $R = G(0) + M^{-1}$, and $\tilde{G}_{ra}(s)$ is the augmented system (49) reduced by the integrator states and has the state space representation (48). This leads directly to the Popov indirect form, with a parallel combination of the augmented system $\tilde{G}_{ra}$ and the constant dynamics $s^{-1}R$, as depicted in figure 10(*b*).

In the passivity framework, stability requires either the feedforward or feedback operator be strictly passive. In this case, strict passivity is achieved by conditions on the reduced system $\tilde{G}_{ra}$ and strict positivity of R. The necessary dissipation for the parallel system is ultimately guaranteed by the existence of some $\Delta > 0$. Naturally, the scalar analogy for the positivity condition: $R = R^{\mathrm{T}} > 0$ on the integrator term is the simple capacitor, which is passive[10] provided the capacitance value is positive. The notion that a linear system can be strictly passive even though it has zero eigenvalues is not intuitive, but similar results are available in the literature, and usually involve decomposing the system into its stable and constant dynamic components, as is done here for the indirect Popov criterion. In Anderson and Vongpanitlerd (1973; p. 216), for example, it is shown that systems with purely imaginary poles are positive real only if the associated residue matrices are non-negative definite Hermitian. A similar state space diagonalization is used to establish Lyapunov stability criteria in Boyd *et al.* (1994; pp. 20–22) for systems having eigenvalues with a zero real part. In essence, Theorem 1 is an extension of these ideas to a particular version of the KYP Lemma, and in effect could be called the *Indirect Control KYP Lemma*, for the historical reasons cited in § 1.

Of course, as is well known, a linear system is strictly passive if and only if its Hermitian form is strictly positive definite for all frequencies (Desoer and Vidyasagar 1975; p. 174); that is, a system $H(s)$ is strictly passive if and only if, for some $\delta > 0$

$$H(j\omega) + H^*(j\omega) > \delta I, \qquad \forall \omega \geq 0 \qquad (51)$$

Hence, the stability question can be addressed by asking the equivalent question: When is a square, linear system having zero eigenvalues strictly passive? Note that, unlike the approach taken in Haddad and Kapila (1995) and Park *et al.* (1998), we do not require the linear system to be strictly positive real (SPR) (Wen 1988, Khalil 1995), which is a stronger condition than strict passivity. In fact, the transformed system $\tilde{G}_a$ in general cannot be SPR since the multiplier $W(s)$ introduces a zero eigenvalue (see Khalil 1996, pp. 404–405); however, it is clear that $\tilde{G}_a$ satisfying the conditions of Theorem 1 are strictly passive. This follows since

$$G_a(j\omega) + \tilde{G}_a^*(j\omega) = \tilde{G}_{ra}(j\omega) + \tilde{G}_{ra}^*(jw) + \frac{1}{j\omega}(R - R^{\mathrm{T}}) \qquad (52a)$$

$$= \tilde{G}_{ra}(j\omega) + \tilde{G}_{ra}^*(j\omega) \qquad (52b)$$

$$> \Delta \qquad (52c)$$

$$\geq \delta I \qquad (52d)$$

where δ is the minimum eigenvalue of Δ. Therefore the strict passivity condition (51) is achieved. Here gain, as in the Theorem 1, the role of symmetric R is apparent, this time in the frequency domain.

7. Numerical examples

7.1. *Computing the maximum allowed slope of non-linearities*

A common engineering problem that often arises is that of finding the maximum sloped non-linearity that a given system can tolerate before going unstable. This problem was posed in Park *et al.* (1998), and an LMI solution was suggested based on the analysis given in that paper. The same problem in terms of the conditions of Theorem 1 is stated as

$$\max \mu \quad \text{subject to:} \quad \begin{cases} (38),\ (39) \\ R = R^{\mathrm{T}} > 0 \end{cases} \qquad (53)$$

where $M = \mu I_m$. Solving (53) for the arbitrary 2×2 system $G(s)$ given as

$$G_1(s) = \begin{bmatrix} \dfrac{s^2 - 0.2s + 0.1}{s^3 + 2s^2 + 2s + 1} & \dfrac{s^2 - 0.4s + 0.75}{s^3 + 3s^2 + 3s + 1} \\ \dfrac{0.1s^2 + 5s + 0.75}{s^3 + 1.33s^2 + 2s + 1} & \dfrac{0.15(s^2 + s + 0.75)}{s^3 + s^2 + 1.1s + 1} \end{bmatrix} \qquad (54)$$

by using the LMI solver (Gahinet *et al.* 1995), yields a maximum allowed slope value of $\mu = 0.940$. By comparison, the equivalent problem using the stability criteria from Park *et al.* (1998) results in a maximum slope of 0.392, approximately a factor of 2 smaller. Obviously, Theorem 1 is less conservative in guaranteeing stability for this system. The reason for this is that while $G(0)$ is symmetric, and thus satisfies the criteria in Park *et al.* (1998), $G(0)$ is a full matrix. As a result of Park's additional constraint, $NG(0) - G(0)^{\mathrm{T}}N = 0$, the multiplier N must reduce to a scalar, positive number. By contrast, our Theorem 1 poses no such condition on $G(0)$, and allows N to remain a diagonal matrix with two degrees of freedom, and is thus able to give less conservative stability guarantees. This relative advantage is likely to increase as

[10] Assuming the input/output relation across the capacitor terminals is current/voltage.

the number of non-linearities increases in the case of non-diagonal $G(0)$. This follows since Theorem 1 will allow one additional degree of freedom for each non-linearity, while the criteria from Park *et al.* (1998) restricts the multiplier to a single scalar number (i.e. $N = nI_m$) regardless of the problem size.

As a second example, consider

$$G_2(s) = \begin{bmatrix} \dfrac{s-0.2}{s^3+2s^2+2s+1} & \dfrac{0.1s^2+1}{s^2+3s+1} \\ \dfrac{0.1s^2+5s+1}{s^2+1.33s+1} & \dfrac{0.2(s^2+s+0.75)}{s^3+s^2+1.1s+1} \end{bmatrix} \quad (55)$$

and note the state space version of this system has a non-zero feedthrough term, $D \neq 0$, and the system matrix at $s = 0$

$$G_2(0) = \begin{bmatrix} -0.2 & 1.0 \\ 1.05 & 0.15 \end{bmatrix}$$

has a negative eigenvalue. For either of these reasons, the recent results of Park *et al.* (1998) and Haddad and Kapila (1995) do not apply in this case. Within the context of absolute stability then, it is fair to conclude the criteria in Park *et al.* (1998) and Haddad and Kapila (1995) can guarantee stability only for non-linearities having *zero* maximum slope (i.e. only when non-linearities are not present). However, Theorem 1 does apply and guarantees stability for all non-linearities in the classes described in § 2 that have a maximum slope less than 0.996. The corresponding stability multiplier is $N = \text{diag}(25.327, 11.134)$.

7.2. *Asymptotic stability with single hysteretic relay*

As a simple example of an application of Theorem 1 for a system with a single non-linearity, consider a third order system

$$G(s) = \frac{s^2+0.01s+0.25}{(s+1)(s^2+s+1)} \quad (56)$$

that is attached in negative feedback with a hysteretic relay (figure 3). A simple graphical check, as described in § 4.2 shows that the line $\phi = -y/G(0)$ intersects the non-linearity in two stable points, $\phi = \pm 1$, and does not intersect the discontinuous portion of the characteristic. In this case the stationary set is well defined and, according to definition (35), is simply two discrete points in state space

$$\mathsf{E} = \left\{ \pm \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\} \quad (57)$$

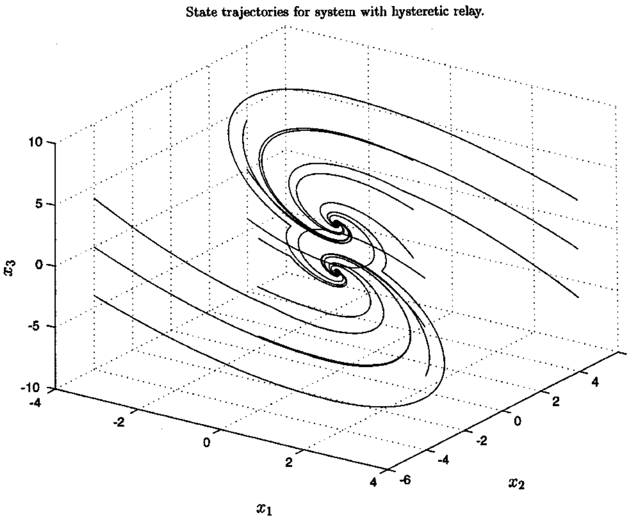


Figure 11. All solutions converge to the two points in the set E.

To prove asymptotic stability of E, we solve the LMI (39) by approximating the infinite slope of the relay with the value $\mu = 1 \times 10^6$. Using the LMI toolbox (Gahinet *et al.* 1995), the stability

$$P = \begin{bmatrix} 5.0826 & -0.02149 & 0.16304 \\ -0.02149 & 4.7911 & -0.02991 \\ 0.16304 & -0.02991 & 3.038 \end{bmatrix}$$

$N = 4.7078$, and $\Delta = 4.77 \times 10^{-6}$, which proves the global asymptotic stability of the set E. Note that in this case $G(s)$ is not positive real, and thus an analysis of this hysteretic relay system based on the circle criteria, such as the IQC technique given by Rantzer and Megretski (1996), would fail. However, the graph of $G(j\omega)$, $w \geq 0$ does not enter the third quadrant of the Nyquist plane and therefore satisfies less restrictive stability criteria for systems with scalar hysteresis non-linearities, as detailed in Paré and How (1998 b). Several simulations of the non-linear system confirm this result.

The set is clearly visible in figure 11, as initial conditions at various locations in state space converge to either of the two discrete points. The non-linear behaviour of the system is evident in figure 12, which shows non-smooth trajectories of the state that result at times when the relay switches. The non-linear switching is also the cause of the asymmetric pattern of the state strajectories, as seen in the x_1–x_3 plane.

7.3. *Asymptotic stability with multiple backlash non-linearities*

Here we investigate the stability of the two-input, two-output system

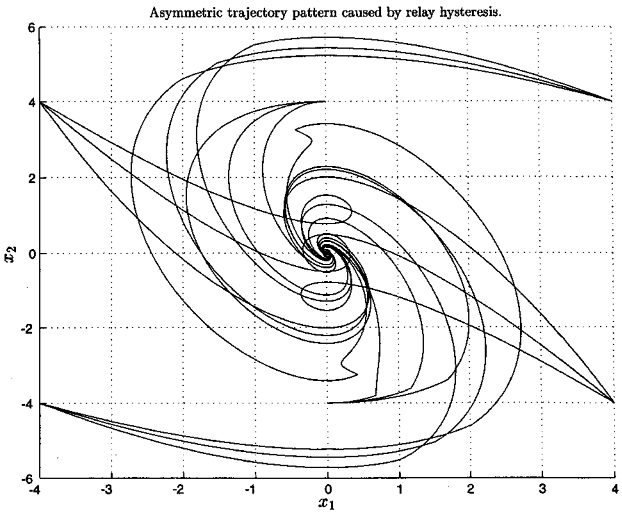


Figure 12. Trajectories are non-smooth as a result of relay switching.

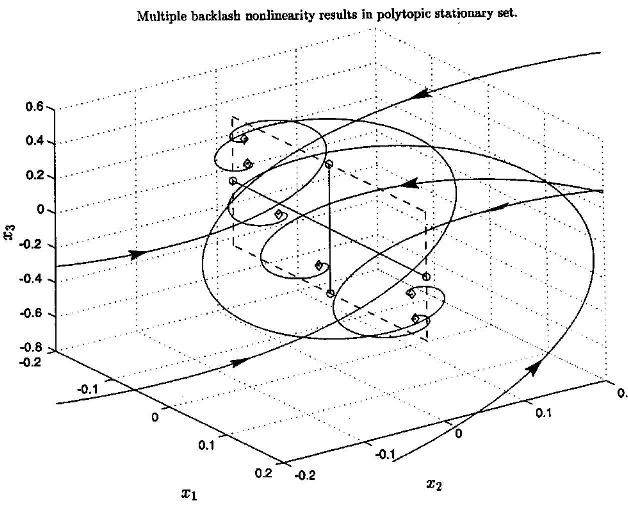


Figure 13. The stationary set E for a multiple backlash non-linearity is a rectangular region in the x_1–x_3 plane.

$$G \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

$$= \left[\begin{array}{ccc|cc} -2 & -1 & -0.5 & 0.19365 & 0.41312 \\ 2 & 0 & 0 & 0 & -0.41312 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 1.875 & -0.1875 & 0.09375 & 0 & 0 \\ 1 & 0.75 & 1 & 0 & 0 \end{array}\right] \tag{58}$$

that is attached in feedback with two backlash non-linearities, described in figure 4, each having unit slope and deadband width ($\mu, D = 1$). The system matrix at $s = 0$

$$G(0) = \begin{bmatrix} 0.0363 & 0.3873 \\ 0.3873 & 0.20656 \end{bmatrix}$$

is symmetric, and has eigenvalues $\lambda = -0.275$, 0.518, so that the criteria $R = R^{\mathrm{T}} > 0$, where $R = G(0) + I$ is satisfied. Solving the LMI (39) yields the stability parameters

$$P = \begin{bmatrix} 4.2914 & -1.921 & -3.7638 \\ -1.921 & 7.6573 & -2.3389 \\ -3.7638 & -2.3389 & 18.354 \end{bmatrix}$$

and

$$N = \begin{bmatrix} 1.7292 & 0 \\ 0 & 1.6253 \end{bmatrix} \quad \Delta = \begin{bmatrix} 0.75697 & -0.18976 \\ -0.18976 & 0.69497 \end{bmatrix}$$

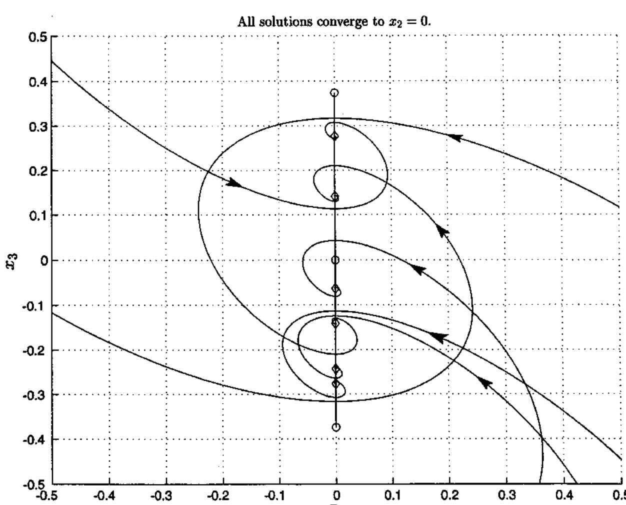


Figure 14. As viewed in the x_2–x_3 plane, the set E appears as a segment of the x_3-axis.

Positivity of these matrices proves global asymptotic stability for the set E, as per Theorem 1. In this case, E is a polytopic region, as described for the backlash non-linearity by equation (37), given by

$$\mathsf{E} = \mathbf{Co}\left\{ \pm \begin{bmatrix} 0.1712 \\ 0 \\ 0 \end{bmatrix}, \pm \begin{bmatrix} 0 \\ 0 \\ 0.3737 \end{bmatrix} \right\} \tag{59}$$

The stationary set E (59) is shown dashed in figure 13. Simulation of the nonlinear system with six different initial conditions confirms the stability of the set. All trajectories terminate in E, as shown in figure 13. The perspective looking down onto the x_2–x_3 plane, given in figure 14, confirms that the second component of the state indeed converges to zero, since the various trajectories all end in the corresponding segment of the x_3-axis.

8. Conclusions

This paper establishes absolute stability criteria for systems with multiple hysteresis and slope-restricted non-linearities. Using Popov's indirect control form as an analytical framework, a Lyapunov stability proof is developed to guarantee stability for these two classes of non-linear systems. The analysis for the two different cases is effectively unified by introducing a transformation that converts either non-linearity into a passive operator. In the hysteresis case, the Lyapunov function includes an integral term that is dependent on the non-linearity input–output path, while the corresponding Lyapunov term for the memoryless non-linearity is not. As a result of the new analysis, early work performed by Yakubovich for a scalar hysteresis is extended to handle multiple non-linearities, and recent work on multiple slope-restricted non-linearities is further generalized. The stability guarantee is presented in terms of a simple linear matrix inequality (LMI) in the given matrices, and a certain residue matrix condition that must be satisfied. Asymptotic stability is with respect to a subset of state space that contains all equilibrium positions of the non-linear system. Descriptions of these stationary sets for several common hysteresis types are given in detail. Simple numerical examples are then used to demonstrate the effectiveness of the new analysis in comparison to other recent results, and graphically illustrate state asymptotic stability. By contrast to the previous work, our analysis allows for non-strictly proper systems and, except for trivial cases such as a diagonal system matrix, the stability multiplier is allowed to be more general and leads to less conservative stability predictions.

9. Convergence limit proof

In this section, the existence of $\lim_{t\to\infty}\eta(t)$ is established. This is done first by showing that $\dot\eta\in\mathcal{L}_1$, and then by employing the Lebesgue Monotone Convergence theorem. Recall first, as a result of the Lyapunov proof, and the differentiability of the non-linearity, that $\dot\eta\in\mathcal{L}_\infty$. We note that the dynamics of the transformed system in figure 7 can be equivalently depicted as resulting from an external input signal that is the initial condition response of the (open loop) linear subsystem

$$u_2(t) = y_h(t) + R\eta(0)$$

where $y_h(t) = CA^{-1}\,e^{At}\,x(0)$, as is shown in figure 15. The initial state of the linear subsystem, $\tilde G_r(s)+(1/s)I$, is then considered zero, and its output

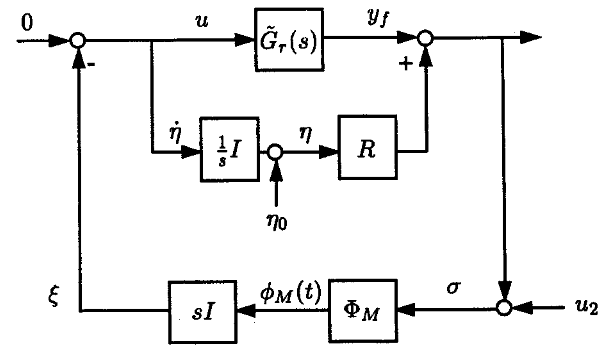


Figure 15. Popov system with initial condition response as input.

$$y_1(t) = y_f(t) + R\eta(t)$$

with $y_f(t) = CA^{-1}\int_0^t e^{A(t-\tau)}\,Bu(\tau)\,d\tau$. We then have

$$\dot\eta = -\frac{d}{dt}\Phi_M(\sigma(t)) \qquad (60\,a)$$

$$= -\Phi'_M(\sigma)\frac{d}{dt}\sigma(t) \qquad (60\,b)$$

$$= -\Phi'_M(\sigma)\frac{d}{dt}\{y_1+u_2\} \qquad (60\,c)$$

$$= -\Phi'_M(\sigma)\frac{d}{dt}\{y_f+R\eta+y_h+R\eta(0)\} \qquad (60\,d)$$

where $\Phi'_M(\sigma)$ is the diagonal matrix of local slopes occurring at the m scalar non-linearities

$$\Phi'_M(\sigma) = \text{diag}\,\{\phi'_i(\sigma_i),\ldots,\phi'_m)\} > 0 \qquad (61)$$

Inserting the identities

$$\dot y_f(t) = C\int_0^t e^{A(t-\tau)}B\dot\eta(\tau)\,d\tau + CA^{-1}B\dot\eta(t)$$

and

$$R\dot\eta(t) = (-CA^{-1}B + D + M^{-1})\dot\eta(t)$$

into (60 d) results in

$$\dot\eta = -\Phi'_M[(G+M^{-1})\dot\eta + C\,e^{At}x(0)] \qquad (62)$$

where $G\colon e\mapsto y$ is original system operator (as depicted in figure 6(a), M^{-1} is the diagonal matrix of maximum slopes and, for simplicity of notation, the dependence on σ is dropped. Solving for $\dot\eta$ gives

$$\dot\eta(t) = -\{I+\Phi'_M(G+M^{-1})\}^{-1}\Phi'_M C\,e^{At}\,x(0) \qquad (63)$$

where the inverse exists since $\dot\eta$ is bounded. That is, the input–output mapping

$$\mathcal{G} = \{I+\Phi'_M(G+M^{-1})\}\Phi'_M \qquad (64)$$

is $\mathcal{L}_\infty$ stable. Designating the peak gain[11] as $\|\mathcal{G}\|_{\infty,i}<\infty$, $\dot\eta$ can be bounded pointwise in time with c_1, $c_2>0$, as

[11] Sometimes called *induced* $\mathcal{L}_\infty$-norm of the operator.

$$|\dot{\eta}(t)| \leq \|\mathcal{G}\|_{\infty,i} |C\,\mathrm{e}^{At}\,x(0)| \quad (65\,a)$$

$$\leq c_1 \|\mathcal{G}\|_{\infty,i} \mathbf{1}\,\mathrm{e}^{-c_2 t} \quad (65\,b)$$

where

$$c_1\,\mathrm{e}^{-c_2 t} \geq \max_k \{\sup_{t \geq 0} |\dot{y}_{h,k}(t)|\}$$

$k = 1, \ldots, m$, with $\dot{y}_h(t) = C\,\mathrm{e}^{At} x(0)$ and $\mathbf{1} \in \mathbf{R}^m$ is a vector of 1's. In particular, $-c_2$ is the real part of the 'slowest' eigenvalue of A (assumed Hurwitz, so that $c_2 > 0$ is guaranteed), and c_1 is chosen simply to ensure the exponential envelope bounds all m elements of $|\dot{y}_h(t)|$, $t \geq 0$. Therefore, the following holds

$$\|\dot{\eta}\| = \sum_{k=1}^{m} \int_0^\infty |\dot{\eta}_k(t)|\,\mathrm{d}t \quad (66\,a)$$

$$\leq m \max_k \int_0^\infty |\dot{\eta}_k(t)|\,\mathrm{d}t \quad (66\,b)$$

$$\leq m c_1 \|\mathcal{G}\|_{\infty,i} \int_0^\infty \mathrm{e}^{-c_2 t}\,\mathrm{d}t \quad (66\,c)$$

$$= m \frac{c_1}{c_2} \|\mathcal{G}\|_{\infty,i} \quad (66\,d)$$

$$< \infty \quad (66\,e)$$

Thus, $\dot{\eta} \in \mathcal{L}_1$, which ensures the existence of $\lim_{t\to\infty} \int_0^t \dot{\eta}(\tau)\,\mathrm{d}\tau$.[12] Therefore, $\eta(t)$ asymptotically becomes constant. Furthermore, in this case, convergence to that constant is exponential.

[12] A straightforward way to see this is to write $\int_0^t \dot{\eta}(\tau)\,\mathrm{d}\tau$ as a Lebesgue integral and perform the standard decomposition into positive/negative sequences, $\int_{[0,t]} \dot{\eta}\,\mathrm{d}\lambda = \int_{[0,t]} \dot{\eta}^+\mathrm{d}\lambda - \int_{[0,t]} \dot{\eta}^-\mathrm{d}\lambda$. Since both $\int_{[0,t]} \dot{\eta}^+\mathrm{d}\lambda$ and $\int_{[0,t]} \dot{\eta}^-\mathrm{d}\lambda$ are monotonically increasing and bounded, the limit is guaranteed by the Lebesgue Montone Convergence theorem (see Rudin 1987, Oden and Demkowicz 1996, for example, or any real analysis text).

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