Proceedings of the 38th
Conference on Decision & Control
Phoenix, Arizona USA • December 1999

Algorithms for Reduced Order Robust $\mathcal{H}_\infty$ Control Design

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Abstract

While there has been much work done in recent years on robust control synthesis, most frameworks produce controllers that are full order. This paper presents LMI-based algorithms that explicitly synthesize robust controllers that have reduced order. That is, the controller transfer functions have fewer poles than the plant. This reduction is accomplished by treating the controller order as part of a multi-objective optimization. As expected, the reduced controllers typically achieve degraded performance as the order is reduced. This trade-off is explored in this paper using three design approaches: $\mathcal{H}_\infty$, robust $\mathcal{H}_\infty$, and Popov/$\mathcal{H}_\infty$ design techniques, and the results are presented using trade off curves for a typical benchmark problem. The reduced order design algorithms are useful in the cases when robust performance is critical but the controller order must be constrained due to limitations of control hardware, or excessive order of the plant.

1 Introduction

The popularity of linear matrix inequalities as a framework to analyze the stability of uncertain systems and to design linear robust controllers has grown rapidly over the last five years. Linear matrix inequalities allow the user the freedom to express diverse concepts such as Lyapunov stability, dissipation theory, passivity and energy gain all in a single compact notational form [1]. Recently LMI's have found extensive use in $\mathcal{H}_\infty$ multi-objective control design [2, 3, 4, 5]. Most of this work involves full-order design, whereby the controllers designed have the same order as the plant. However, it is commonly reported in these works that the order of controller is tied directly to the rank of a certain positive matrix of the form:

$$M = \begin{bmatrix} R & I \\ I & S \end{bmatrix} \tag{1}$$

that occurs in the design process. In essence, minimum order control design requires solving for positive matrices, R and S, that simultaneous minimize the rank of M and satisfy the set of constraints that guarantee a preselected performance level. Unfortunately, while the performance constraint set is convex, the minimum rank set is not, and therefore the joint problem is not suitable for current LMI software [6, 7]. Several attempts have been made to overcome this problem. In [8], for example, the eigendecomposition of the nonconvex constraint is used to produce an approximate subgradient for a descent direction in an optimization routine. It is noted however, that this technique is numerically cumbersome in practice, and the nondifferentiability of the constraints often leads to convergence problems. A more recent approach utilizes an alternating projection algorithm [9, 10, 11] which produces minimum order controllers by projecting the two matrices onto the convex and nonconvex constraint sets. As reported in Ref. [11], the alternating projection scheme is often not necessary in practice, and that minimizing the linear objective $c = Trace(M)$ often leads to the minimum order stabilizing controller. This possible connection between the rank of a matrix having the form (1) and its trace is explored in recent preliminary work by Mesbahi [12]. Essentially, Ref. [12] points out that positive matrices of the form (1) that satisfy the closed loop stability constraints form a type of set referred to as a hyperlattice, and minimum rank elements in this set will also have minimum trace (see also [13]).

In this paper we use this observation about $Trace(M)$ as the basis to generate reduced order controllers. In particular, we treat the $Trace(\cdot)$ function as a *convex relaxation* of the $Rank(\cdot)$ objective to develop LMI based algorithms to synthesize controllers that optimize an $\mathcal{H}_\infty$ performance objective subject to an order condition on the controller. First, we incorporate the trace objective into a bisection algorithm to perform a Pareto optimal investigation that trades off controller order for performance. We then extend the design objective to include a robustness constraint in addition to performance. For this case, the objective function has two parts, one part pertaining to performance and the other for control order. Finally, we present an algorithm for reduced order Popov/$\mathcal{H}_\infty$ controllers. Using a three-mass benchmark problem, we show that this algorithm consistently produces reduced order controllers that give better performance than those designed using the robust $\mathcal{H}_\infty$ algorithm.

The paper is organized as follows. In §2, three reduced order synthesis problem statements for $\mathcal{H}_\infty$, robust $\mathcal{H}_\infty$ and Popov/$\mathcal{H}_\infty$ controllers, are detailed. These problems are followed in §3 with the corresponding LMI design algorithms. A brief discussion is given in §4 to compare the new design approach to a balanced real reduction technique for a typical benchmark problem; conclusions are then provided in §5.

2 Synthesis Problem Statements

The design for the controller $K(s)$ that will achieve a closed loop $\mathcal{L}_2$-gain across some performance channel using an LMI framework is well documented in the literature [2, 3, 4, 5]. All

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presentations start with the bounded-real lemma [1, p. 23], and use the Elimination and Completion Lemmas to convert the problem into an optimization over a set of convex constraints which is readily solved using available LMI solvers [7, 6]. Here we use those well established techniques to arrive at the equivalent convex optimization problem. The typical problem, depicted in Fig. 1, is to design a proper, linear controller:

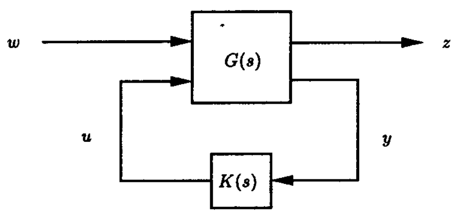

Fig. 1: Standard $\mathcal{H}_\infty$ control design set-up.

$$K \stackrel{s}{=} \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right], \tag{2}$$

that will achieve an $\mathcal{L}_2$-gain of γ from input disturbance w to performance output z of the linear plant $G(s)$, given as:

$$G \stackrel{s}{=} \left[\begin{array}{c|c|c} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ \hline C_y & D_{yw} & 0 \end{array}\right]. \tag{3}$$

2.1 H_∞ Control

Following the results of [2, 3, 4, 5], there exists a controller of the form (2) achieving the $\mathcal{H}_\infty$ upper bound γ if and only if there exist positive definite matrices R and S such that the conditions:

$$U^T \left[\begin{array}{cc|c} AR+RA^T & RC_z^T & B_w \\ C_zR & -\gamma I & D_{zw} \\ \hline B_w^T & D_{zw}^T & -\gamma I \end{array}\right] U \;<\; 0 \tag{4a}$$

$$V^T \left[\begin{array}{cc|c} A^TS+SA & SB_w & C_z^T \\ B_w^TS & -\gamma I & D_{zw}^T \\ \hline C_z & D_{zw} & -\gamma I \end{array}\right] V^T \;<\; 0, \tag{4b}$$

are satisfied, with

$$U = \left[\begin{array}{c|c} \begin{bmatrix} B_u \\ D_{zu} \end{bmatrix}^\perp & 0 \\ \hline 0 & I \end{array}\right], \text{ and } V = \left[\begin{array}{c|c} \begin{bmatrix} C_y^T \\ D_{yw}^T \end{bmatrix}^\perp & 0 \\ \hline 0 & I \end{array}\right]$$

and the matrices R, S related by:

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \;\geq\; 0 \tag{5a}$$

$$\text{Rank}\begin{bmatrix} R & I \\ I & S \end{bmatrix} \;\leq\; n+n_c. \tag{5b}$$

The matrix pair (R,S) that satisfies (4) is a convex set (this follows simply because the set is linear in R,S). Following the notation in Ref. [11] we refer to this convex set simply as:

$$\mathbf{\Gamma}_{convex} = \{\ (R,S) \mid R,S \in \mathbb{S}^n,\ \ (4a\text{–}b)\ \}. \tag{6}$$

However, as noted, the set described by the rank condition (5) is non-convex, and will be referred to by:

$$\mathcal{Z}_{n_c} = \{\ (R,S) \mid R,S \in \mathbb{S}^n,\ \ (5a\text{–}b)\ \}. \tag{7}$$

Therefore, there exists a controller of the form (2) that achieves a given performance level γ, if and only if there exists a pair of matrices $R,S > 0$ such that $(R,S) \in \mathbf{\Gamma}_{convex} \bigcap \mathcal{Z}_{n_c}$. The optimal control problem is now stated as:

H_∞ Control Problem. Find the matrix pair R,S that solves the optimization problem:

$$\begin{array}{rl} \min: & \gamma \\ \text{such that:} & R,S>0,\ \ (R,S) \in \mathbf{\Gamma}_{convex} \bigcap \mathcal{Z}_{n_c}. \end{array} \tag{8}$$

The pair R,S that solves (8) completely parameterizes the optimal H_∞ controller (2).

An algorithm which solves this reduced order $\mathcal{H}_\infty$ control problem (8) is detailed in §3.1.

2.2 Robust H_∞ Control

With this concise statement for H_∞ control, we consider now the robust control problem in which the plant has uncertainty Δ, as is depicted in the standard three-block system set-up shown in Fig. 2. As is common practice in control theory [14], the optimal control design for the uncertain plant is achieved using either the bounded real lemma (small gain), or the Popov criterion. In this framework, we seek a linear controller $K(s)$ for a plant G_α that has norm-bounded uncertainty captured by the Δ-block, where $\Delta \in \mathbf{\Delta}$, with

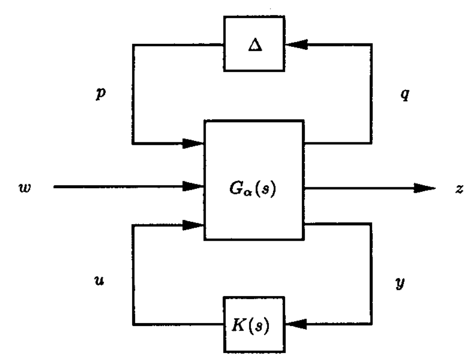

Fig. 2: Standard Robust $\mathcal{H}_\infty$ control design set-up.

$$\mathbf{\Delta} = \{\Delta \in RH_\infty : \|\Delta\| < 1\}. \tag{9}$$

The performance channel is then scaled by a constant α that is used to determine the achievable performance of the closed loop system. The scaled system is defined as:

$$G_\alpha \;\stackrel{s}{=}\; \left[\begin{array}{c|cc|c} A & B_p & \sqrt{\alpha}B_w & B_u \\ \hline C_q & D_{qp} & \sqrt{\alpha}D_{qw} & D_{qu} \\ \sqrt{\alpha}C_z & \sqrt{\alpha}D_{zp} & \alpha D_{zw} & \sqrt{\alpha}D_{zu} \\ \hline C_y & D_{yp} & \sqrt{\alpha}D_{yw} & 0 \end{array}\right] \tag{10a}$$

$$\triangleq \left[\begin{array}{c|c|c} A & B_w^\alpha & B_u \\ \hline C_z^\alpha & D_{zw}^\alpha & D_{zu}^\alpha \\ \hline C_y & D_{yw}^\alpha & 0 \end{array}\right], \tag{10b}$$

where we artificially combine the uncertainty and scaled performance channels into a single channel. Making the substitutions, $B_w^\alpha \leftarrow B_w$, *etc.*, in the set of inequalities (4), we define a new convex set $\mathbf{\Gamma}_{convex}^\alpha$ in terms of the scaled inequalities. Thus, we say there exists a robust controller that stabilizes the system G for all uncertainties $\Delta \in \mathbf{\Delta}$ if and only if there exists a pair of matrices $R, S > 0$ such that $(R,S) \in \mathbf{\Gamma}_{convex}^\alpha \bigcap \mathcal{Z}_{n_c}$, with the parameter fixed at $\gamma = 1$. We can now state the robust H_∞ control problem.

Robust $\mathcal{H}_\infty$ Control Problem. Find the matrix pair R, S that solves the optimization problem:

$$\begin{array}{rl} \max: & \alpha \\ \text{subject to:} & (R,S) \in \mathbf{\Gamma}_{convex}^\alpha \bigcap \mathcal{Z}_{n_c},\ \gamma = 1. \end{array} \tag{11}$$

The pair R, S that solves (11) parameterizes the controller that achieves optimal, robust H_∞ performance. The optimal performance is $1/\alpha$, and is guaranteed for all $\Delta \in \mathbf{\Delta}$.

The algorithm which produces robust $\mathcal{H}_\infty$ controllers, detailed in problem (11), is given in §3.2.

2.3 Popov/H_∞ Control

This robust design approach in §2.2 is known to result in conservative controllers if the plant uncertainty can be modeled as a sector-bounded, memoryless nonlinearity. This occurs, for instance, when the plant G includes a real parameter with a value only known to within a certain tolerance. A better approach in such cases would be to use the Popov stability criterion, which can be much less restrictive than the small gain constraint [15]. Robust performance for a system with uncertainty $\Delta \in$ sector$[0,1]$, using this criterion is guaranteed by the LMI [1, p. 122]:

$$\begin{bmatrix} \tilde{A}^T P + P\tilde{A} + \tilde{C}_z^T \tilde{C}_z & L_{12} & P\tilde{B}_w \\ L_{12}^T & L_{22} & \Lambda\tilde{C}\tilde{B}_w \\ \tilde{B}_w^T P & \tilde{B}_w^T \tilde{C}^T \Lambda & -\gamma^2 I \end{bmatrix} \leq 0, \tag{12}$$

where $L_{12} = P\tilde{B}_p + \tilde{A}^T\tilde{C}_q\Lambda + \tilde{C}^T T$, $L_{22} = \Lambda\tilde{C}B_p + B_p^T\tilde{C}^T\Lambda - 2T$ and $\Lambda, T \in R^{n_q \times n_q}$ are diagonal, non-negative and referred to as the stability multipliers. The system matrices in (12) are assumed to represent the closed loop dynamics depicted in Fig. 2. Assuming a full-order controller, as given by (2), the same approach used to formulate the H_∞ constraint set results in the following inequalities that guarantee existence of the Popov/H_∞ controller:

$$U_\perp \left[\begin{array}{ccc|c} AR + RA^T & M_{12} & RC_z^T & B_w \\ M_{12}^T & M_{22} & 0 & \Lambda C_q B_w \\ C_z R & 0 & -\gamma I & D_{zw} \\ \hline B_w^T & B_w^T C_q^T \Lambda & D_{zw}^T & -\gamma I \end{array}\right] U_\perp^T < 0 \tag{13a}$$

$$V_\perp^T \left[\begin{array}{ccc|c} A^T S + SA & N_{12} & SB_w & C_z^T \\ N_{12}^T & N_{22} & \Lambda C_q B_w & 0 \\ B_w^T S & B_w^T C_q^T \Lambda & -\gamma I & D_{zw}^T \\ \hline C_z & 0 & D_{zw} & -\gamma I \end{array}\right] V_\perp < 0, \tag{13b}$$

where $M_{12} = B_p + RA^T C_q^T \Lambda + RC_q^T T$, $M_{22} = \Lambda C_q B_p + B_p^T C_q^T \Lambda - 2T$, $N_{12} = SB_p + A^T C_q^T \Lambda + C_q^T T$, $N_{22} = \Lambda C_q B_p + B_p^T C_q^T \Lambda - 2T$ and the outer matrices are

$$U_\perp = \left[\begin{array}{c|c} \begin{bmatrix} B_u \\ \Lambda C_q B_u \\ D_{zu} \end{bmatrix}^\perp & 0 \\ \hline 0 & I_{n_w} \end{array}\right], \quad V_\perp^T = \left[\begin{array}{c|c} \begin{bmatrix} C_y^T \\ D_{yp}^T \\ D_{yw}^T \end{bmatrix}^\perp & 0 \\ \hline 0 & I_{n_z} \end{array}\right]. \tag{14}$$

These constraints have the same form as those in (4). Note that these inequalities are again linear in the pair R, S, and in the multiplier matrices Λ, T. However, the set is not jointly linear in both pairs of variables since products of R with Λ and T appear in the $(1,2)$ term of the constraint (13a). Subsequently, Eqn. (13) is considered a bilinear matrix inequality (BMI), and is thus not convex jointly in the two sets of variables. The common approach (see [15], and references therein) is to consider convex subsets that result when fixing either the pair R, S or the multipliers Λ, T. Fixing Λ, T gives the set:

$$\mathbf{\Gamma}(\Lambda, T) = \{\ (R,S)\ |\ \text{(13a–b)}\ \}; \tag{15}$$

and, similarly, holding R, S fixed defines the convex set:

$$\mathbf{\Gamma}(R,S) = \left\{\ (\Lambda, T)\ \middle|\ (12),\ \ P = \begin{bmatrix} S & I \\ I & (S - R^{-1})^{-1} \end{bmatrix}\ \right\}. \tag{16}$$

Note, that for $\mathbf{\Gamma}(R,S)$ it is assumed that the feasible Popov/H_∞ controller corresponding to the given R, S is used to form the closed loop system matrices in the Popov LMI (12). For convenience, define the set of diagonal $m \times m$ non-negative matrices as D_+^m. Now the quadruples that satisfy the sets above can be used to define a Popov solution. Specifically, let $\mathbf{\Gamma}_{Popov} : \mathsf{S}^n \times \mathsf{S}^n \times \mathsf{D}_+^m \times \mathsf{D}_+^m$ be defined as:

$$\mathbf{\Gamma}_{Popov} = \left\{\ (R,S,\Lambda,T)\ \middle|\ \begin{array}{l} (R,S) \in \mathbf{\Gamma}(\Lambda,T) \\ (\Lambda,T) \in \mathbf{\Gamma}(R,S) \end{array}\ \right\}. \tag{17}$$

We can now state the Popov/H_∞ control problem as an optimization problem similar to the H_∞ and robust H_∞ cases. However, unlike these previous cases, the Popov/H_∞ is not a convex optimization problem since, as alluded to, the constraint set $\mathbf{\Gamma}_{Popov}$ is not convex.

Popov/H_∞ Control Problem. There exists a controller that robustly stabilizes the system if there exists a quadruple $(R,S,\Lambda,T) \in \mathbf{\Gamma}_{Popov}$. The minimum γ that can be achieved for any such quadruple is the optimal Popov/H_∞ solution. This leads to the following optimization problem:

$$\begin{array}{rl} \min & \gamma \\ \text{subject to:} & \left\{ \begin{array}{l} (R,S,\Lambda,T) \in \mathbf{\Gamma}_{Popov} \\ (R,S) \in \mathcal{Z}_{n_c} \end{array} \right. \end{array} \tag{18}$$

An algorithm to solve (18) is detailed in §3.3.

3 Algorithm Descriptions

How do you put these three control problems into practice? In particular, for a desired order of controller, what are the algorithms that can be used to numerically implement these problems that will find the optimal reduced order controllers? We answer these questions with a series of algorithms in this section. Naturally, they range in complication from easiest to hardest, with the Popov/H_∞ requiring the most sophistication.

Note that in each case, the algorithms produce a pair of matrices (R,S) that completely parameterize the reduced order solution. To obtain the controller K from a pair (R,S) that solves one of the optimization problems requires the solution of a feasibility problem:

$$M_\gamma(R,S) + UKV^T + VK^TU^T < 0 \tag{19}$$

where U, V are complements of the corresponding matrices above, and $M_\gamma(R,S)$ is a constant matrix involving the open loop system matrices, the specified performance level, and the pair (R,S). These matrices are easily derived using the elimination technique well documented in [2] and omitted here for brevity.

We illustrate each algorithm using a benchmark three mass-spring system (see appendix for details). The benchmark system is characterized by a rigid body mode and two flexible modes, one of which is non-colocated. By including a small amount of damping in between the masses, the minimum order stabilizing controller is known in advance to be first order. This simply corresponds to a first order lead network required to stabilize the pole pair at the origin. We can expect however, that any controller designed to be first order will not perform very well due to the presence of the lightly damped, non-colocated mode.

3.1 Reduced order $\mathcal{H}_\infty$ design

Solving for reduced H_∞ controllers is straightforward, and we present one approach that utilizes a simple bisection search that is a direct extension to the work in [11, 12]. Specifically, a five-step iteration that finds the best fixed-order H_∞ controller is as follows:

i) Set upper, lower performance bounds, convergence tolerance: γ_u, γ_l, and *tol*
ii) Set desired controller order: n_{c-des}
iii) Repeat {

1. $\gamma = \frac{1}{2}(\gamma_U + \gamma_L)$
2. *Solve:* min: $\mathrm{Tr}(R+S)$; subject to: $(R,S) \in \Gamma_{convex}$
3. Decompose matrix: $[U^T, \Sigma, U] = svd(R - S^{-1})$
4. $n_c = length(diag(\Sigma))$
5. if $n_c > n_{c-des}$, $\gamma_l = \gamma$, else $\gamma_u = \gamma$}

until: $(\gamma_u - \gamma_l) < tol * \gamma_l$

This algorithm produces a controller of prescribed order that gives the best H_∞ performance. It performs a bisection, or line search, in order to determine the best performance possible for the given controller order. The upper and lower bounds on the search are adjusted according to whether or not a feasible controller exists that achieves the particular performance level γ. The order of the controller is equal to the number of non-zero elements of the singular value decomposition ($svd(\cdot)$) of the matrix $R - S^{-1}$ (see [2, 3, 4] for discussion). Of course, the accuracy of the search can be adjusted using the tolerance level *tol*.

For the benchmark problem, the six controllers (five reduced and one full order) were generated using this algorithm. The corresponding maximum singular value curves are depicted in Fig. 3, along with the bar graph which summarizes the performance as a function of controller order, shown in Fig. 4. Clearly, as expected, the achievable performance reduces with controller order. Controller orders of five and six both achieve the $\mathcal{H}_\infty$ limit of $\gamma = 3.7$, and have flat σ_{max} curve typical of a full-order optimal controller. However, as the controllers are restricted to fourth order and lower, peaks appear in the σ_{max} curves since the reduced order controllers can not completely notch the flexible body modes; lower order controllers result in higher peaks. This relationship between controller order and performance is captured by the the bar graph in Fig. 4. In a sense, the bar graph Fig. 4 can be considered a Pareto optimal curve that depicts the trade-off between two competing design objectives.

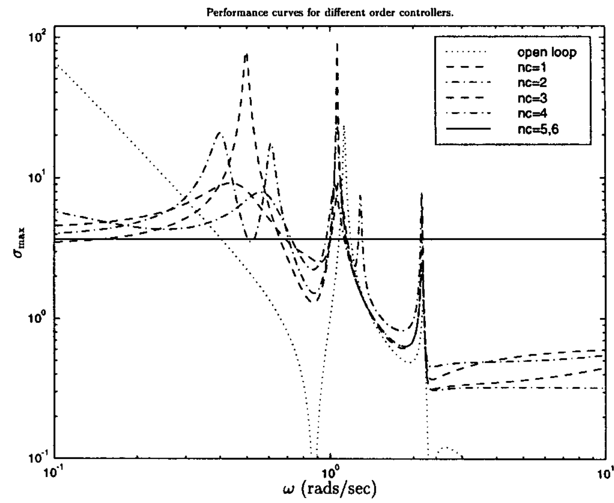

Fig. 3: Reduced order designs for benchmark problem; Maximum singular values, $\sigma\mathrm{max}(\omega)$.

3.2 Reduced order robust $\mathcal{H}_\infty$ control

The design for robust H_∞ controllers requires only a slight modification to the previous algorithm. Here the bisection is on the performance channel weighting α, and not the control order explicitly. The objective function now consists of two parts: one part affecting controller order, $Trace(R+S)$ and the other performance γ. The same trade-off between order and performance is achieved by varying the relative weight between the two cost components.

The algorithm is as follows:

i) Set performance channel bounds, convergence tolerance: α_u, α_l, and *tol*
ii) Set desired relative weight: $\beta \in [0,1]$
iii) Repeat {

1. $\alpha = \frac{1}{2}(\alpha_U + \alpha_L)$
2. *Solve:* $\left\{ \begin{array}{rl} \text{min:} & \beta\gamma + (1-\beta)\mathrm{Tr}(R+S) \\ \text{subject to:} & (R,S) \in \Gamma_{convex} \end{array} \right.$
3. if $\gamma > 1$, $\alpha_u = \gamma$, else $\alpha_l = \alpha$}

until: $(\alpha_u - \alpha_l) < tol * \alpha_l$.

Once again, using the benchmark problem, and now assuming that the spring constant between the second and third masses

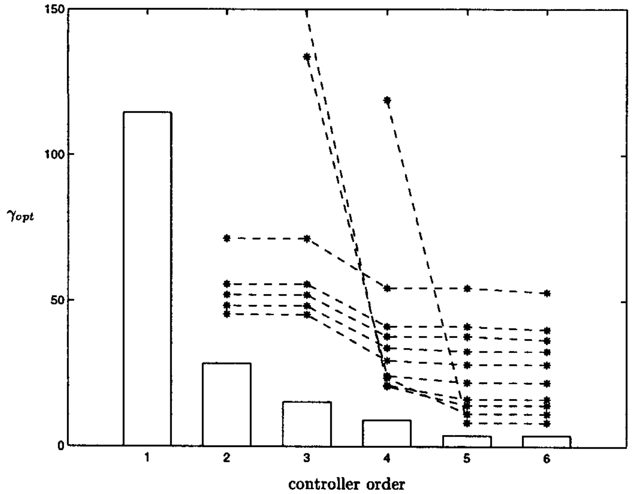

Fig. 4: Comparing reduced order $\mathcal{H}_\infty$ performance to controllers reduced via balanced real reduction. Bar graph: output from reduced order algorithm §3.1; Dashed: results from balanced order reduction.

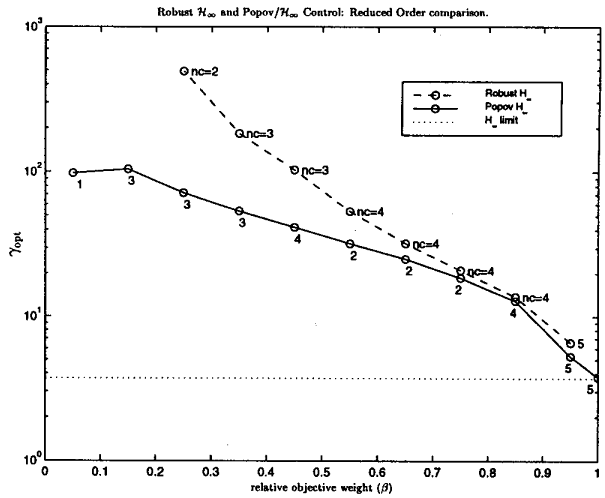

Fig. 5: Reduced order Popov controllers consistently outperform Robust $\mathcal{H}_\infty$ designs.

is only know to within 5%, reduced order controllers were designed using the reduced order algorithm for a range of β values. The resulting performance over the range of β is depicted by the dashed line in Fig. 5. As expected, as β increases, more weight is put on the performance in the cost objective, so that performance improves. Of course, the cost improvement comes at the expense of increased controller order. The graph also depicts the controller order that corresponds with each performance value. Note that the order increases monotonically with β, along with the performance improvement. As $\beta \to 1$, the performance tends toward the non-robust H_∞ limit value of 3.7. It is interesting that a first order controller could not be found with this algorithm. This is most probably due to the conservativism of the small gain stability guarantee. Essentially, this suggests that a 5% change in the spring constant is too much variation to guarantee stability using the small gain theorem when limited to first order controllers.

3.3 Reduced order Popov/H_∞ control

The Popov/H_∞ algorithm is slightly more complicated than the previous two cases. The cost objective once again has two parts, but each iteration involves two optimization steps. The first step optimizes over the controller matrices, while the second step optimizes with respect to the multipliers. The algorithm is summarized by:

i) Initialize controller $K_1(s)$ and multipliers (Λ, T)
ii) Set desired relative weight: $\beta \in [0,1]$
iii) Repeat {

1. *Solve:* $\left\{ \begin{array}{rl} \text{min:} & \beta\gamma + (1-\beta)\text{Tr}(R+S) \\ \text{subject to:} & (R,S) \in \Gamma(\Lambda, T) \end{array} \right.$

2. Find feasible controller, $K_k(s)$, form closed loop matrices

3. *Solve:* min: $\beta\gamma + (1-\beta)\text{Trace}(P)$; subject to (12) }

until: $|\gamma_k - \gamma_{k-1}| < tol * \gamma_k$.

This algorithm was executed for a range of β and the performance with the corresponding controller orders are shown withthe solid curve in Fig. 5, along with the robust H_∞ results. There are several interesting aspects to this data. First, as we might expect, the performance curve for the Popov controllers lies strictly below that for the robust H_∞ designs. This follows since the Popov constraint is less conservative when the uncertainties are assumed to be sector bounded and memoryless. Also, the performance of the controllers monotonically improves as the performance weight β increases, just as the previous case. However, the controller order does not strictly increase as the performance improves. For example, there are several second order controllers for $\beta = 0.55 - 0.75$ that give better performance than the third and fourth order controllers that were produced at lower values of β. It is difficult to attribute this observation to anything other than the known non-convexity of the sets over which the optimization is performed.

Note also that a first order controller is produced using the Popov algorithm that has better performance than the non-robust H_∞ controller. This seems to be an indication that while using the $Trace(\cdot)$ objective might be a good convex approach to minimizing rank, it is still just an approximation. This is evident here since, even though the Popov optimization has the additional stability constraint, the routine was able to find a lower minimum.

4 Discussion

In order to assess the effectiveness of the new algorithms in generating optimal, reduced order controllers, we compare our results to another reduced order design approach. Depicted in Fig. 4 are the same results of the reduced order $\mathcal{H}_\infty$ design algorithm along with the performance of several other reduced order sub-optimal controllers (shown dashed). These sub-optimal cases were designed by taking full-order LQG controllers and reducing the order by means of a balanced realization order reduction. LQG designs were chosen for comparison because the resulting H_∞ norm could be varied systematically above the optimal limit by varying the control and performance weightings. Note that in all cases, controllers that resulted from the balanced reduction technique could not approach the optimal performance levels of the reduced order algorithm from §3.1. As the controllers are truncated, the performance degrades in each case, and the corresponding performance lines all lie above the bar graph. For instance, reducing a full order controller with a performance of $\gamma = 25$, can yield a second order controller with a $\gamma = 47$; however, the new algorithm produces a second order controller with $\gamma = 23$, which is an improvement of about 50%. Pushing the performance of the full order LQG controllers in an attempt to achieve better low order controllers leads to dramatic degradation in γ-levels. In fact, no stabilizing first order controller could be found using the balanced realization reduction scheme.

We note that our search for good low order controllers using the balanced real, order reduction technique was by no means exhaustive. Indeed, there may exist controllers that give better performance than those indicated by the bar graph. However, finding the *true* optimal reduced order controllers is still an open (non-conxex) design problem; the algorithms presented here are an efficient, convex approach towards that goal.

5 Conclusions

Three different LMI-based algorithms for producing reduced order H_∞ controllers are presented. All three utilize the $Trace(\cdot)$ objective function as a means to constrain the order of the controller. One algorithm simply optimizes the H_∞ performance subject to an explicit constraint on the order. The other algorithms produce reduced order robust controllers using a two part objective involving the $Trace(\cdot)$ and the closed loop performance. Using the combined objective, the designer is then free to select the relative weighting on the two parts of the objective cost in order to trade-off performance versus controller order. In this way, this paper provides a valuable tool that allows control designers to perform a multi-objective design analysis in the practical situations when performance is critical and the order of the controller must be reduced because of either real-time control hardware limitations or the excessive order of the plant.

The $Trace(\cdot)$ is used here as a convex relaxation to the rank condition and enables efficient design of reduced order controllers. Also, the reduced order BMI algorithm for the design of Popov/H_∞ controllers is more reliable than previous fixed order design techniques [15] since the new algorithm systematically eliminates poorly conditioned subspaces of matrices which could otherwise lead to numerical instability in the reconstruction of the controller.

6 Acknowledgement

The authors greatly appreciate the preprint copy of Ref. [12] from Dr. Mehran Mesbahi, which provided the insight and motivation for this work.

7 Appendix: Benchmark problem

A typical mechanical benchmark problem for robust control design is depicted in Fig. 6; this problem was used to demonstrate the reduced order algorithms. For this system it is assumed the third spring constant k_3 has 10% uncertainty, and we would like to control the system by applying a force

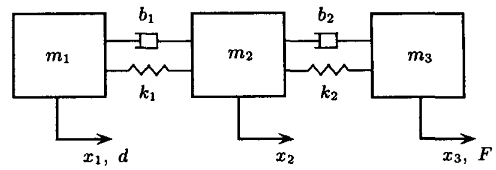

Fig. 6: Benchmark three mass used for algorithm comparisons.

F to the third mass, m_3 using a position measurement, x_1 of the first mass, while the first mass is subject to a disturbance force d. For this system, the performance variable was chosen as $z = [z_1 \;\; z_2]^T$, where $z_1 = x_1 + x_2$ corresponds to the average position of the first two masses, and $z_2 = u$ is the control force. The system parameters were chosen to be $m_1 = m_2 = m_3 = 1.0$, $b_1 = b_2 = 0.015$, and $k_1 = k_2 = 1.0$. Note that the addition of a small amount of damping to the mass/spring system will mean that the minimum order controller for this system is simply first order, corresponding to a stabilization of the rigid body mode.

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