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Synthesizing Stability Regions for Systems with Saturating Actuators

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Abstract

This paper presents a method of synthesizing controllers for systems with input saturation that guarantees state space regions of attraction. While the computation of stability regions and the corresponding state feedback design has appeared recently in the control literature, the more realistic case of output feedback has not been addressed. This note provides a simple design technique using an LMI framework to produce controllers that maximize the region of stability for systems having limited control when only partial state information is available for measurement.

1 Introduction and Problem Statement

This paper extends the analysis work done in [1] to the case of synthesizing output feedback controllers that maximize regions of attraction. In particular, the controllers are designed to maximize the region in state space such that any initial condition x_0 starting in this region will imply $x(t) \to 0$. The controllers G_c will be dynamic, of the same order as the given linear plant, and computed using an LMI approach. The stability regions will not be invariant, but will have a property referred to *pseudo*–invariance [2], which means that state trajectories originating in the region may exit but will eventually return as the state converges. This region is defined by the ellipse

$$\mathcal{E}_P = \{\ x \mid x^T P x < 1\ \} \tag{1}$$

for some $P = P^T > 0$. The plant $G(s)$ is considered LTI with open loop dynamics

$$\begin{aligned} \dot{x} &= Ax + Bp, \qquad x(0) = x_0 \\ y &= Cx \\ p &= sat(u) \end{aligned} \tag{2}$$

The system matrix A is assumed to have unstable eigenvalues. Similarly, G_c has linear dynamics given by

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c \end{aligned} \tag{3}$$

The design objective is find (A_c, B_c, C_c) which maximizes the volume the state space region for which initial conditions of the plant will be guaranteed to converge absolutely to zero.

2 Synthesis Procedure

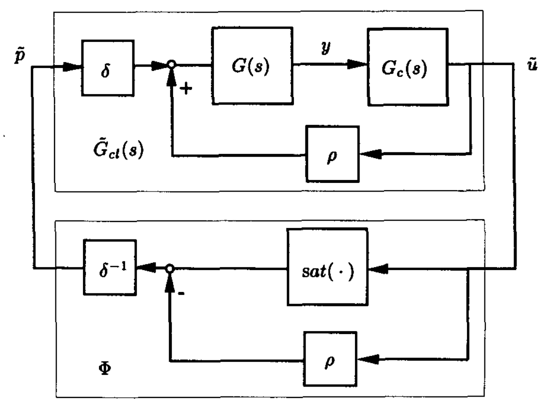

Fig. 1: System with loop transformation.

To solve the maximization problem from the previous section, we follow the work done in [1], which developed analytical means of determining local regions of stability for systems with saturation. The loop transformation shown in Figure 1 is introduced so that local regions can be parameterized by the saturation level r. By defining the constants in the transformation as

$$\rho = \frac{1}{2}(1 + \frac{1}{r}), \quad \delta = \frac{1}{2}(1 - \frac{1}{r}) \tag{4}$$

then the transformed saturation operator, $\Phi : \tilde{u} \to \tilde{p}$ can be thought of as "locally" sector bounded. That is $\Phi \in \text{sect}[-1, 1]$ when $|\tilde{u}(t)| < r$ for all t. Similarly, the transformed system $\tilde{G}_{cl} : \tilde{p} \to \tilde{u}$, has a state space representation in terms of δ, ρ given by

$$\tilde{G}_{cl} \stackrel{s}{=} \left[\begin{array}{c|c} \begin{bmatrix} A & \rho B C_c \\ B_c C & A_c \end{bmatrix} & \begin{bmatrix} \delta B \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & C_c \end{bmatrix} & 0 \end{array}\right] = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & 0 \end{array}\right]. \tag{5}$$

Now, since the $\|\Phi\|_{\mathcal{L}_2} \leq 1$, if we can enforce $\|\tilde{G}_{cl}\|_\infty < 1$, then by the small gain theorem we can conclude that the closed loop system is $\mathcal{L}_2$–stable for the given level of r. If $V(x, x_c) = \tilde{x}^T \tilde{P} \tilde{x}$, $\tilde{P} > 0$ is a storage function for $\tilde{G}_{cl}$

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then $\|\tilde{G}_{cl}\|_\infty < 1$ if and only if $\tilde{G}_{cl}$ satisfies the small gain dissipation inequality $\dot{V} < \tilde{p}^T\tilde{p} - \tilde{u}^T\tilde{u}$, in which case the system is stable and $\tilde{x} = [x^T \; x_c^T]^T \to 0$. The stability constraint imposed by the dissipation has the matrix form

$$\begin{bmatrix} -\tilde{P}\tilde{A} - \tilde{A}^T\tilde{P} - \tilde{C}^T\tilde{C} & -\tilde{P}\tilde{B} \\ -\tilde{B}^T\tilde{P} & I \end{bmatrix} > 0. \quad (6)$$

Approximating the inverse of the ellipse volume as Trace $\tilde{P}$, then, as shown in [3, pg 48], the stability region is maximized by solving

$$\begin{array}{ll} \text{minimize} & \text{Trace}\tilde{P} \\ \text{subject to} & \tilde{P} > 0, \; \tilde{C}_c\tilde{P}^{-1}\tilde{C}_c^{\,T} < r^2, \; (6) \end{array} \quad (7)$$

which is an optimization problem bilinear in the stability parameter $\tilde{P}$ and the controller matrices (A_c, B_c, C_c). To convert this problem to an LMI, we parameterize $\tilde{P}$ and its inverse $\tilde{Q} = \tilde{P}^{-1}$ as

$$\tilde{P} = \begin{bmatrix} P & M \\ M^T & R \end{bmatrix}, \; \tilde{Q} = \begin{bmatrix} Q & N \\ N^T & S \end{bmatrix}. \quad (8)$$

and then apply the Elimination lemma [3] to rid A_c from (6), which yields the equivalent constraints

$$\begin{bmatrix} -A^TP - PA - ZC - C^TZ^T & -\delta PB \\ -\delta B^TP & I \end{bmatrix} > 0 \quad (9a)$$

$$\begin{bmatrix} -AQ - QA^T - \rho BY - \rho Y^TB^T & -\delta B & Y^T \\ -\delta B^T & I & 0 \\ Y & 0 & I \end{bmatrix} > 0 \quad (9b)$$

Similarly, using the Completion lemma, the first two constraints in (7) are expressed as the LMI

$$\begin{bmatrix} P & I & 0 \\ I & Q & Y^T \\ 0 & Y & r^2 \end{bmatrix} > 0. \quad (10)$$

Problem (7) then becomes

$$\begin{array}{ll} \text{minimize} & \text{Trace}P \\ \text{subject to:} & P > 0, Q > 0, (9), (10) \end{array} \quad (11)$$

which is a convex optimization problem with linear matrix inequality constraints, and therefore solveable using available LMI software [4]. The solution to (11) provides matrix variables P, Q, Y and Z which can then be used to reconstruct the controller G_c, as described in [2]. This controller maximizes the state space stability region for G, as defined by (1).

3 Numerical Example: Balanced Pointer

Here we consider an example using (11) to maximize the region of attraction $\mathcal{E}_P$ for a simple inverted pendulum system. The dynamics of the unstable plant $G(s)$ are described by

$$A = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (12)$$

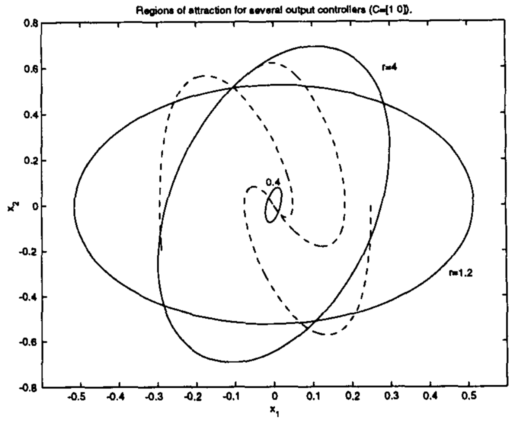

Fig. 2: Guaranteed regions of attract. for various r values.

and $C = [\; 1 \quad 0 \;]$, which is not stabilizable using static output feedback. The optimal regions of attraction were solved for values of $r = 0.4$, 1.2. and 4.0, and are plotted in Figure 2. The region is quite small for $r = 0.4$ (strictly linear operatation), but increases for larger values of r. The controller for $r = 4$ was used for a nonlinear time simulation of several initial condition responses. Several initial condition response trajectories (dashed lines) confirm that the attraction region is not invariant, as two of the state trajectories leave the region, but we have $x(t) \to 0$ as $t \to \infty$.

4 Conclusions

This paper extends recent analysis of systems with saturation with an LMI-based procedure to synthesize dynamic controllers that maximize the region of attraction for systems that have saturating actuators. The design procedure presented produces output feedback controllers that guarantee maximum regions of attraction while allowing the system to run at a prescribed level of saturation. The algorithm is used to compute controllers for a simple inverted pendulum system to effectively produce stability regions several times larger than that guaranteed for a controller restricted to operate in the linear region.

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