Convergence Analysis of A Parametric Robust $\mathcal{H}_2$ Controller Synthesis Algorithm[1]

David Banjerdpongchai
Durand Bldg., Room 110
Dept. of Electrical Engineering
Email: banjerd@isl.stanford.edu

Jonathan P. How[2]
Durand Bldg., Room 277
Dept. of Aeronautics and Astronautics
Email: howjo@sun-valley.stanford.edu

Stanford University, Stanford CA 94305

Abstract

This paper presents an iterative algorithm for solving the parametric robust $\mathcal{H}_2$ controller synthesis problem and analyzes the convergence properties of the algorithm on several examples. Iterative procedures are normally applied to a large class of robust control design problems in which the formulation naturally leads to bilinear matrix inequalities (BMIs). It is difficult to make concrete statements about the behavior of these iterative algorithms, except that it is often conjectured that the cost in each step of the solution procedure is reduced, which implies that the algorithms should converge to a local minimum. Similar difficulties exist for the new LMI-based iterative algorithm that we have recently proposed to solve the BMIs that occur in robust $\mathcal{H}_2$ control design. The effectiveness of the new algorithm has already been demonstrated on several numerical examples. This paper adds an important component to the discussion on the convergence of the new algorithm by verifying that it efficiently converges to the optimal solution. In the process, we provide some new key insights on the proposed design technique which indicate that it exhibits properties similar to the D–K iteration of the complex μ/K_m-synthesis.

1 Introduction

The *bilinear matrix inequality (BMI)* approach has been demonstrated to be effective for solving the complex/real μ/K_m synthesis problems [1, 2]. The global optimization of BMIs is NP-hard, and it is unlikely that there is a polynomial time algorithm to compute the optimal solutions [3, 4]. However, Goh *et al.* [5, 6] devised algorithms for solving the BMI problem via a straightforward method, *i.e.*, using the currently available LMI tools, such as [7, 8, 9, 10]. The approach is to alternately minimize the performance cost subject to BMI constraints with respect to some variables where the other variables are fixed, and vice versa. They have demonstrated that the iterative technique based on the BMI framework improves the guaranteed lower bounds to multivariable stability margins of the closed-loop system by 10% over the corresponding results from the *D–K iteration* with no increase in the controller order. A key advantage of the BMI technique is that it enables control engineers to address several open problems of the robust control synthesis, namely the complex/real μ/K_m synthesis via dynamical scalings or multipliers, the fixed order control synthesis, and the decentralized controller architecture. El Ghaoui and Balakrishnan [11] have concurrently proposed a similar approach using the numerical optimization in the synthesis of fixed-structure controllers and it has been demonstrated to work well on simple examples. However, on complicated objectives, such as control designs to minimize an $\mathcal{H}_2$ cost function, the iterative technique has been found to converge very slowly, if at all. Furthermore, a thorough study of the convergence behavior of the iterative algorithms in Refs. [5, 6, 11] remains to be done in order to better understand the effectiveness and efficiency of these approaches.

The robust controller synthesis algorithm was improved by El Ghaoui and Folcher leading to a more systematic design approach for systems with unstructured uncertainty [12] and structured uncertainty [13]. The design objectives considered include robust stability, robust $\mathcal{H}_2$ performance, robust settling time, and robust input peak bound. A heuristic algorithm [14], which is a local optimization, is used to solve these BMI problems. The algorithm produces dynamic, output-feedback controllers of the order equal to or smaller than the order of the nominal system. However, the single quadratic Lyapunov function used in Refs. [12, 13, 14] can be a source of significant conservatism in the robust control design for systems with real parametric uncertainty.

We have extended the design procedure in Refs. [12, 13] to develop a new synthesis algorithm for systems with parametric uncertainty. This controller synthesis combines the robust stability analysis with Popov multipliers and robust performance bounds on the total output

[1] This research was supported by Ananda Mahidol Foundation and in part by AFOSR under grant F49620-95-1-0318.

[2] Author to whom all correspondence should be addressed. Tel: (650) 723-4432; Fax: (650) 725-3377.

energy for systems subject to sector bounded nonlinear uncertainty [15]. An extension of this synthesis that incorporates generalized multipliers to capture real parametric uncertainty is discussed in [16].

We also take advantage of the closed-loop system structure to eliminate some design parameters from the problem formulation and use an iterative procedure to calculate the remaining variables. Exploiting the structure of design parameters in the synthesis formulation appears to significantly improve the convergence of the algorithm. The main result of this paper is to numerically verify that the iterative algorithm presented in Ref. [15] efficiently converges to the optimal solution. In the process, we compute a global optimum for this BMI problem by an exhaustive search. While the exhaustive search is not particularly efficient, this process is similar in many ways to the proposed global BMI optimizations [5, 6] and provides a simple check of the iterative solution.

2 Problem Statement

We consider the Lur'e system described by

$$\begin{aligned} \dot{x} &= Ax + B_p p + B_w w + B_u u \\ q &= C_q x + D_{qp} p + D_{qw} w + D_{qu} u \\ z &= C_z x + D_{zp} p + D_{zw} w + D_{zu} u \\ y &= C_y x + D_{yp} p + D_{yw} w + D_{yu} u \\ p &= \phi(q), \end{aligned} \tag{1}$$

where $x : \mathbf{R}_+ \to \mathbf{R}^n$ is the state, $u : \mathbf{R}_+ \to \mathbf{R}^{n_u}$ is the control input, $w : \mathbf{R}_+ \to \mathbf{R}^{n_w}$ is the disturbance input, $y : \mathbf{R}_+ \to \mathbf{R}^{n_y}$ is the measured output, $z : \mathbf{R}_+ \to \mathbf{R}^{n_z}$ is the performance output, $q : \mathbf{R}_+ \to \mathbf{R}^{n_p}$ and $p : \mathbf{R}_+ \to \mathbf{R}^{n_p}$ are the input/output of the nonlinear uncertainty ϕ. The nonlinear perturbation ϕ is assumed to satisfy the sector bound $[0, 1]$, *i.e.*, $\phi \in \mathbf{\Phi}$ where

$$\mathbf{\Phi} := \left\{ \begin{array}{l} \phi : \mathbf{R}^{n_p} \to \mathbf{R}^{n_p}, \\ \phi(q) = [\phi_1(q_1), \ldots, \phi_{n_p}(q_{n_p})]^T, \\ \text{where } 0 \le \phi_i(\sigma)/\sigma \le 1,\ \forall\, i = 1, \ldots, n_p. \end{array} \right\}.$$

The description of the Lur'e system also includes an important class of uncertain systems described by

$$\begin{aligned} \dot{x} &= (A + \Delta A)x + B_w w + B_u u \\ z &= C_z x + D_{zw} w + D_{zu} u \\ y &= C_y x + D_{yw} w + D_{yu} u \\ \Delta A &\in \mathcal{U}, \end{aligned} \tag{2}$$

where

$$\mathcal{U} := \left\{ \begin{array}{l} \Delta A \in \mathbf{R}^{n \times n} : \Delta A = B_p D C_q, \\ D = \mathbf{diag}(\delta_1, \ldots, \delta_{n_p}), \\ \text{where } \delta_i \in [0,\, 1],\ \forall\, i = 1, \ldots, n_p. \end{array} \right\}.$$

In control theory, (2) is referred to as the system subject to *real parametric uncertainty* [17, 18]. This special case of the Lur'e system occurs when the functions ϕ_i are linear, *i.e.*, $\phi_i(\sigma) = \delta_i \sigma$, where $\delta_i \in [0,\, 1]$, $\forall\, i = 1, \ldots, n_p$. For well-posedness, we will assume that D_{zw} is identically zero, and to significantly simplify the analysis and synthesis, we assume D_{zp}, D_{qp}, D_{qw}, and D_{qu} are identically zero. We are interested in finding a strictly proper full order LTI controller of the form

$$\dot{x}_c = A_c x_c + B_c y, \quad u = C_c x_c,$$

where $x_c : \mathbf{R}_+ \to \mathbf{R}^n$ is the controller state, A_c, B_c, and C_c are constant matrices of appropriate size. The design objective is to guarantee the robust stability and minimize an upper bound of the worst case $\mathcal{H}_2$ performance. As discussed in Ref. [15], by applying the Popov robust $\mathcal{H}_2$ performance analysis to the closed-loop Lur'e system, the control design problem can be formulated as a non-convex optimization over the variables $\tilde{P}$, Λ, T, A_c, B_c and C_c, *i.e.*,

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr} \tilde{B}_w^T \left[\tilde{P} + \tilde{C}_q^T \Lambda \tilde{C}_q \right] \tilde{B}_w \\ \text{subject to} & \tilde{P} > 0,\ \Lambda \ge 0,\ T \ge 0, \end{array}$$
$$\begin{bmatrix} \tilde{A}^T \tilde{P} + \tilde{P}\tilde{A} + \tilde{C}_z^T \tilde{C}_z & \tilde{P}\tilde{B}_p + \tilde{A}^T \tilde{C}_q^T \Lambda + \tilde{C}_q^T T \\ \tilde{B}_p^T \tilde{P} + \Lambda \tilde{C}_q \tilde{A} + T\tilde{C}_q & \Lambda \tilde{C}_q \tilde{B}_p + \tilde{B}_p^T \tilde{C}_q^T \Lambda - 2T \end{bmatrix} \le 0, \tag{3}$$

where

$$\left[\begin{array}{c|c|c} \tilde{A} & \tilde{B}_p & \tilde{B}_w \\ \hline \tilde{C}_q & \tilde{D}_{qp} & \tilde{D}_{qw} \\ \hline \tilde{C}_z & \tilde{D}_{zp} & \tilde{D}_{zw} \end{array} \right] = \left[\begin{array}{cc|c|c} A & B_u C_c & B_p & B_w \\ B_c C_y & A_c + B_c D_{yu} C_c & B_c D_{yp} & B_c D_{yw} \\ \hline C_q & 0 & 0 & 0 \\ \hline C_z & D_{zu} C_c & 0 & 0 \end{array} \right].$$

3 Solution Procedure

We note that (3) is a BMI problem, *i.e.*, there are product terms involving the analysis variables ($\tilde{P}$, Λ, and T) and compensator variables (A_c, B_c, and C_c) in the performance objective and constraints. Observing the structure of the compensator parameters in the last matrix inequality in (3), the first step of the design procedure is to eliminate some controller parameters from the problem formulation. We then solve for the remaining variables, and use these results to construct the controllers. An iterative algorithm is used to calculate the compensators, but in the process the procedure capitalizes on the very efficient design tools [7, 8, 9, 10] that are available for solving LMI problems. The resulting compensators are full-order, and cannot include architecture constraints. However, the solution procedure is very robust which reduces the user workload. Furthermore, this approach is easily expandable to include other sophisticated analysis tests such as one in Ref. [16].

3.1 The V–K Iteration

Since the controller matrix A_c only appears in the last matrix constraint in (3), it is possible to reduce the number of variables in the problem by eliminating A_c. Applying the Elimination Lemma [19, page 32], it is not difficult to show that (3) is equivalent to

$$\begin{aligned}&\text{minimize } \mathbf{Tr} \begin{bmatrix} B_w \\ D_{yw} \end{bmatrix}^T \begin{bmatrix} P + C_q^T \Lambda C_q & Z \\ Z^T & X \end{bmatrix} \begin{bmatrix} B_w \\ D_{yw} \end{bmatrix} \\ &\text{subject to } \begin{bmatrix} X & Z^T & 0 \\ Z & P & I \\ 0 & I & Q \end{bmatrix} > 0,\ \Lambda \geq 0,\ T \geq 0, \\ &\begin{bmatrix} F_{11} & F_{12} \\ F_{12}^T & F_{22} \end{bmatrix} < 0, \quad \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12}^T & H_{22} & 0 \\ H_{13}^T & 0 & -I \end{bmatrix} < 0, \end{aligned} \tag{4}$$

where

$$\begin{aligned} F_{11} &= PA + A^T P + ZC_y + C_y^T Z^T + C_z^T C_z, \\ F_{12} &= PB_p + ZD_{yp} + A^T C_q^T \Lambda + C_q^T T, \\ F_{22} &= \Lambda C_q B_p + B_p^T C_q^T \Lambda - 2T, \\ H_{11} &= AQ + QA^T + B_u Y + Y^T B_u^T, \\ H_{12} &= B_p + (QA + Y^T B_u^T) C_q^T \Lambda + QC_q^T T, \\ H_{13} &= QC_z^T + Y^T D_{zu}^T,\ H_{22} = \Lambda C_q B_p + B_p^T C_q^T \Lambda - 2T. \end{aligned}$$

Given that there exist P, Q, Y, Z, X, Λ and T satisfying (4), we can construct a controller by the following steps. First, by the Completion Lemma [20], we compute $\tilde{P}$, which is parameterized by $\tilde{P} = \begin{bmatrix} P & M \\ M^T & M^T(P - Q^{-1})^{-1} M \end{bmatrix}$, where M is an arbitrary invertible matrix. Because M corresponds to a change of coordinates in the controller states, the choice of M has no effect on the controller transfer function [12, 13]. After $\tilde{P}$ is constructed, we proceed by computing the input/output controller matrices (B_c and C_c), which are parameterized by $B_c = M^{-1} Z$ and $C_c = Y(I - PQ)^{-1} M$. With $\tilde{P}$, Λ, T, B_c, and C_c determined, it suffices to find A_c satisfying the last matrix constraint in (3) which can then be formulated as an LMI feasibility problem in A_c, *i.e.*,

$$\text{find } A_c \text{ satisfying } \quad \tilde{G} + V A_c^T U^T + U A_c V^T < 0, \tag{5}$$

where $\tilde{G}$, U, and V are defined as

$$\begin{aligned} \tilde{G} &:= \begin{bmatrix} \tilde{A}_0^T \tilde{P} + \tilde{P} \tilde{A}_0 + \tilde{C}_z^T \tilde{C}_z & \tilde{P} \tilde{B}_p + \tilde{C}_q^T T + \tilde{A}_0^T \tilde{C}_q^T \Lambda \\ \tilde{B}_p^T \tilde{P} + T \tilde{C}_q + \Lambda \tilde{C}_q \tilde{A}_0 & \Lambda \tilde{C}_q \tilde{B}_p + \tilde{B}_p^T \tilde{C}_q^T \Lambda - 2T \end{bmatrix}, \\ V &:= \begin{bmatrix} \tilde{J}^T & 0 \end{bmatrix}^T,\ U := \begin{bmatrix} \tilde{J}^T \tilde{P} & 0 \end{bmatrix}^T, \\ \tilde{A}_0 &:= \begin{bmatrix} A & B_u C_c \\ B_c C_y & B_c D_{yu} C_c \end{bmatrix}, \tilde{J} := \begin{bmatrix} 0 \\ I \end{bmatrix}. \end{aligned}$$

Since there are product terms involving compensator parameters and the Popov parameters in (3) and (4), our approach to solving the non-convex optimization problems is based on an iterative procedure. The proposed algorithm, which we call the *V–K iteration*, basically alternates between three different LMI problems, *i.e.*, (3) with fixed compensator parameters, (4) with fixed multiplier parameters, and (5). The first LMI problem, considered as the *V iteration* or *analysis iteration*, is to solve (3) with fixed compensator parameters (A_c, B_c, and C_c) which yields Popov multiplier parameters (Λ and T). For the *K iteration* or *synthesis iteration*, the second and third LMI problem are solved. The solution parameters of the second LMI problem, *i.e.*, (4) with fixed multiplier parameters, implicitly includes the input/output compensator matrices (B_c and C_c) as variables. After obtaining B_c and C_c, the dynamics of the compensator A_c can be computed by solving the third LMI problem (5). At this point a robust compensator, which guarantees the robust stability and satisfies the upper bound of the worst case $\mathcal{H}_2$ performance, is completely calculated. We then repeat the procedure until satisfying the stopping criterion, such as the decrease in the upper bound of the worst case $\mathcal{H}_2$ performance is less than a given absolute and relative accuracy.

Remark 3.1 This solution procedure is not guaranteed to converge globally, but our experience is that it efficiently converges to a local optimum. We will demonstrate the convergence of the alogithm and compare the iterative and optimal solution in the numerical example in §4. Note that each step of the V–K iteration can be solved very efficiently by a previously developed semidefinite programming algorithm SP [8] and very easily coded using a user-friendly interface SDPSOL [10].

Remark 3.2 There are two important distinctions between the V–K iteration and the D–K iteration of the complex μ/K_m synthesis. First, in our approach there are shared variables between each iteration: specifically, $\tilde{P}$ is the common variable between the V iteration and K iteration (where $\tilde{P}$ appears as P, Q, X, Y and Z). However, for the D–K iteration, the D iteration (the robust analysis with or without curve fitting) is entirely separate from the K iteration (the $\mathcal{H}_\infty$ synthesis). We conjecture that these shared variables play a key role in the efficiency of the convergence of this new algorithm to a local optimum. Second, the new solution procedure also eliminates the curve-fitting of the structured singular value because the multiplier can be parameterized in the synthesis formulation and these multiplier parameters are solved simultaneously with the controller parameters.

Remark 3.3 As a consequence of the systematic solution procedure and concise representation of the robust $\mathcal{H}_2$ control design, this technique can easily be extended to consider the $\mathcal{H}_\infty$ norm, instead of the $\mathcal{H}_2$ cost, for designing parametric robust controllers [21]. In contrast, there appears to be no direct simple extension for the current robust $\mathcal{H}_\infty$ control approaches to include the $\mathcal{H}_2$ cost. This shows a unique versatility of our iterative algorithm for designing robust controllers.

3.2 The Global BMI Optimization

In this subsection, we present a simple global optimization technique for computing the optimal solution of (4). We first note that when the Popov multiplier parameters (Λ and T) are fixed, the optimization problem (4) is simply a semidefinite program in the remaining variables P, Z, Q, Y, and X. Hence, we can find the optimal solution by taking a fine grid of nodes in some closed bounded space of the multiplier parameters, *i.e.*, $\Lambda \times T \in [\Lambda_L, \Lambda_U] \times [T_L, T_U]$, where $0 \le \Lambda_{L,i} < \Lambda_{U,i} < \infty$, and $0 \le T_{L,i} < T_{U,i} < \infty$, $i = 1, \ldots, n_p$. This restriction on the multiplier space is consistent with the assumption required for global optimization techniques of the general BMI problems [6]. The initial space of the multiplier parameters can be specified by letting (Λ_L, T_L) equal to zero, and (Λ_U, T_U) be the values computed from the Popov robust $\mathcal{H}_2$ performance analysis of the closed-loop Lur'e system in which controller parameters are given and fixed. Then, the search is refined to regions of interest that contain local minima.

At each node, we can solve the corresponding semidefinite program and obtain the upper bound of the worst case $\mathcal{H}_2$ performance. As discussed in Ref. [6], the solution of (4) is a locally Lipschitz continuous function over a bounded set of the multiplier parameters. Therefore, the optimal robust performance is the least upper bound of the worst case $\mathcal{H}_2$ performance over these nodes. Since the accuracy of the optimal value depends on the grid size, we must increase the number of nodes in a particular region to obtain a more precise solution. In general, the exhaustive search is not a particular efficient nor elegant method to calculate the optimal solution since the dimension (*i.e.*, the number of nodes) grows exponentially with the number of the nonlinearities. However, in practice this approach provides the simplest available method of solving the global optimization for BMI problems, and thus is very useful for investigating the convergence of the V–K iteration on simple systems.

Remark 3.4 Goh *et al.* [5, 6] have attempted to achieve global solutions of BMI problems via branch and bound algorithms. The lower bounds are computed by using local optimization algorithms and the upper bounds are calculated from a relaxed version of the BMI problems, *i.e.*, by introducing new variables to represent the products of the original variables. This approach has been demonstrated to converge for non-control examples, but to the best of our knowledge, it has not been applied to the very complex robust control design problem discussed in this paper.

4 Numerical Examples

We demonstrate the convergence of the iterative algorithm for the Popov $\mathcal{H}_2$ controller synthesis on the three mass-spring benchmark system [22] (see Figure 1). We first consider the case that the second spring constant has ±5% uncertainty, *i.e.*, $k_2 = k_{2,nom}(1+\delta)$, where $k_{2,nom}$ is the nominal value and $\delta \in [-0.05,\ 0.05]$. The system parameters are $m_1 = m_2 = m_3 = 1$, $k_1 = 1$, and $k_{2,nom} = 1$. To apply this Popov $\mathcal{H}_2$ synthesis technique, we effectively approximate the uncertainty in the spring stiffness as $k_2(x) = k_{2,nom}[x + \sigma\phi(x)]$, where $\sigma = 0.05$

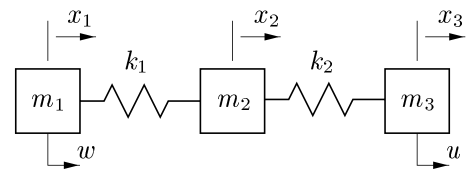

Figure 1: Configuration of the three mass-spring system.

is a measure of the guaranteed uncertainty bound, and $\phi(x)$ is a $[-1,\ 1]$ sector-bounded memoryless nonlinear function of the spring displacement, x. The iterative algorithm in §3.1 is used to compute the robust controller, where the stopping criterion is based on 1% absolute and relative accuracy. The history of the V–K iteration is summarized in Table 1. As shown in the table, the V–K iteration yields the Popov $\mathcal{H}_2$ controller after only four iterations.

Table 1: Summary of the V–K iteration of the Popov $\mathcal{H}_2$ controller designed for the three mass-spring system with ±5% guaranteed uncertainty of the second spring constant.

Iter. #	Up bnd $\mathcal{H}_2$ Cost	% Error $\mathcal{H}_2$ Cost	Multiplier parameters Λ	Multiplier parameters T
0	47.087	168.698	45.358	91.420
1	17.605	0.464	1.437	3.101
2	17.530	0.036	1.478	2.559
3	17.525	0.003	1.474	2.427
Opt	17.524	0	1.474	2.378

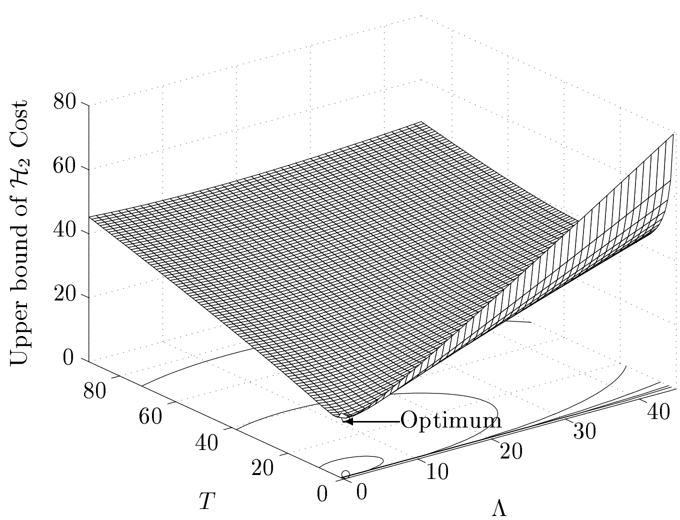

Figure 2: Mesh plots of the $\mathcal{H}_2$ cost overbounds as a function of the multiplier parameters.

As discussed in §3.2, the optimal solution can be computed by first coarsely gridding a closed bounded space of the multiplier parameters, *i.e.*, the set of total 93×47 nodes over the space $\Lambda \times T \in [0,\ 92] \times [0,\ 46]$ with an equal grid size. This region was selected to include the constraints on (Λ, T) and the initial value of the V–K iteration. The surface of the $\mathcal{H}_2$ cost overbound is shown in Figure 2. We then refined the region, particularly near the local optimum to obtain a new set of 41×41 nodes over the subspace $\Lambda \times T \in [0.7,\ 2.2] \times [1.2,\ 3.6]$ with an even grid size. The contour plots shown in Figure 3 and the mesh plots shown in Figure 4 illustrate the worst case $\mathcal{H}_2$ cost overbounds over this refined multiplier space. Note that these plots display only the refined region of the search grid. The first surprising observation seen in these figures is that there is only one local minimum over the entire grid. Furthermore, the worst case $\mathcal{H}_2$ cost overbound behaves like a convex function in this region of the multiplier parameters. In three dimensions, the cost surface shown in Figure 4 looks like smooth paraboloid which has a local minimum at $\Lambda = 1.474$ and $T = 2.378$.

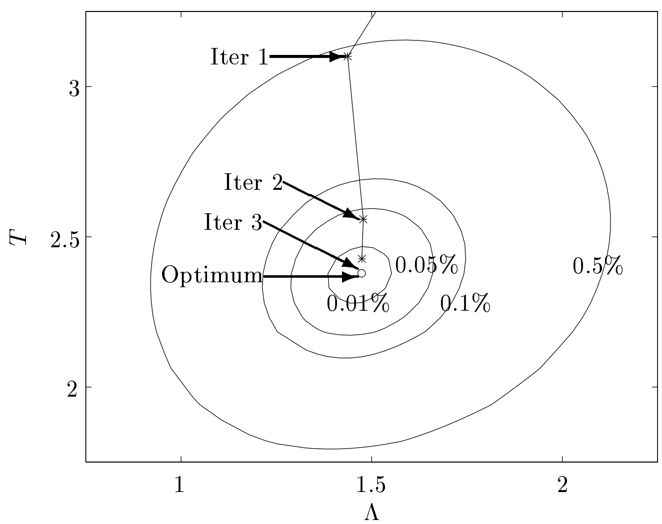

Figure 3: Contour plots of the $\mathcal{H}_2$ cost overbounds as a function of the multiplier parameters over a refined region.

We continue the discussion of the convergence of the V–K iteration by drawing the path connecting the multiplier of each iteration marked by "$*$" in Figure 3 and 4. Note that the optimal Popov multiplier computed by the exhaustive search is marked by "$\circ$" in these figures and also shown in Table 1. Labels next to the contour lines of Figure 3 represent the percentage cost increment with respect to the optimal $\mathcal{H}_2$ cost overbound. The initial multiplier, as shown in the table, lies far outside the region of the contour plot depicted in these figures. However, after the first iteration, the algorithm results in a multiplier for which the worst case $\mathcal{H}_2$ cost overbound is within 0.5% of the optimal value. As shown in this figure, the algorithm stops after the third iteration which corresponds to a worst case $\mathcal{H}_2$ cost overbound within 0.01% of the optimal value. This result clearly shows that in this case, the algorithm converges to the optimal solution.

To further demonstrate that the algorithm converges locally for systems with multiple nonlinearities, we consider the second case of the three mass-spring system where the stiffnesses of both springs are uncertain with $\pm 5\%$ guaranteed bounds. As before, we compute the Popov $\mathcal{H}_2$ controller using the V–K iteration in §3.1. For the stopping criterion of 1% accuracy, four itera-

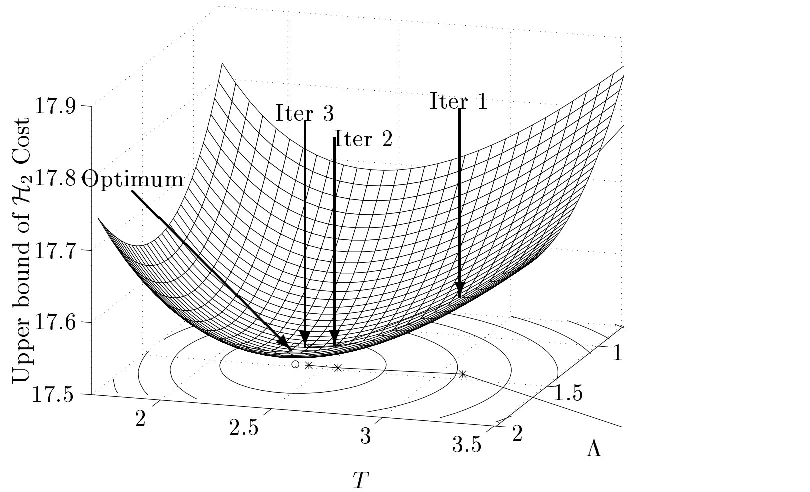

Figure 4: Mesh plots of the $\mathcal{H}_2$ cost overbounds as a function of the multiplier parameters over a refined region.

tions are required to solve for this robust controller. Table 2 summarizes the history of the V–K iteration for the case with two uncertainties. The exhaustive search procedure described in §3.2 was used to calculate the optimal solution for the BMI problem (4). The coarse grid (11^4 nodes) was uniformly taken over the parameter space $\Lambda \times T = \mathbf{diag}(\lambda_1, \lambda_2) \times \mathbf{diag}(\tau_1, \tau_2)$, $\lambda_1 \in [1, 3]$, $\lambda_2 \in [1, 3]$, $\tau_1 \in [1.3, 3.8]$, and $\tau_2 \in [1, 3]$. After carefully observing the location of a local minimum, we refined the search to 9^4 nodes with an even grid size over the parameter region $\Lambda \times T$, $\lambda_1 \in [1.8,\ 2.2]$, $\lambda_2 \in [1.7,\ 2.1]$, $\tau_1 \in [2.3,\ 2.8]$, and $\tau_2 \in [1.8,\ 2.2]$. Of course, there is no convenient visualization tool, so only numerical results are presented here in Table 2. As shown previously in the one uncertainty case, the overbound of the worst case $\mathcal{H}_2$ performance is monotonically decreasing during the iterative solution. Furthermore, at the third iteration, the algorithm yields a solution that is very close to the optimum, *i.e.*, to within 0.01%. Note however that the multiplier parameters from the V–K iteration shown in the table are not as close to the optimal ones in this case.

These convergence results are typical of the trends seen for each Popov $\mathcal{H}_2$ control design presented in Ref. [15]. To the best of our knowledge, this analysis indicates that, for these numerical examples, the V–K iteration has in fact converged to the globally optimal solution.

Table 2: Summary of the V–K iteration of the Popov $\mathcal{H}_2$ controller designed for the three mass-spring system with $\pm 5\%$ guaranteed uncertainty of both spring constants.

Iter. #	Up bnd $\mathcal{H}_2$ Cost	% Error $\mathcal{H}_2$ Cost	Multiplier parameters λ_1, λ_2	τ_1, τ_2
0	19.772	0.79	1.421, 1.316	2.383, 1.872
1	19.650	0.16	1.735, 1.628	2.469, 1.936
2	19.625	0.04	1.883, 1.784	2.499, 1.966
3	19.620	0.01	1.954, 1.859	2.512, 1.978
Opt	19.618	0	2.013, 1.923	2.520, 1.985

5 Conclusions

This paper presents convergence results of the LMI-based iterative algorithm for solving the Popov $\mathcal{H}_2$ controller synthesis problem. Our approach has already been shown to yield consistent Popov $\mathcal{H}_2$ controllers, when compared with compensators designed by quasi-Newton search technique. The numerical results presented here indicate that the V–K iteration converges to the optimal solution in a few iterations and demonstrates the efficiency of the proposed algorithm in practice. In the process, we use a simple exhaustive search to compute the BMI global optimization, but this technique is limited to BMI problems with a small number of uncertainties. The convergence of the V–K iteration confirms our previous design experience and, more importantly, it offers key insights into the design methodology and the benefits of exploiting the structure of the controller matrices. Because of the low overhead associated with developing and implementing the LMI optimization, the V–K iteration can be easily extended to other sophisticated analysis tests and controller constraints.

References

[1] M. G. Safonov, K. C. Goh, and J. H. Ly, "Control System Synthesis via Bilinear Matrix Inequalities," in *Proc. American Control Conf.*, pp. 45–49, 1994.

[2] K. C. Goh, J. H. Ly, L. Turand, and M. G. Safonov, "μ/k_m-Synthesis via Bilinear Matrix Inequalities," in *Proc. IEEE Conf. on Decision and Control*, pp. 2032–2037, Dec. 1994.

[3] O. Toker and H. Özbay, "On the NP-Hardness of Purely Complex μ Computation, Analysis/Synthesis, and Some Related Problems in Multidimensional Systems," in *Proc. American Control Conf.*, pp. 447–451, June 1995.

[4] O. Toker and H. Özbay, "On the NP-Hardness of Solving Bilinear Matrix Inequalities and Simultaneous Stabilization with Static Output Feedback," in *Proc. American Control Conf.*, pp. 2525–2526, June 1995.

[5] K. C. Goh, L. Turan, M. G. Safonov, G. P. Papavassilopoulos, and J. H. Ly, "Biaffine Matrix Inequality Properties and Computational Methods," in *Proc. American Control Conf.*, pp. 850–855, 1994.

[6] K. C. Goh, M. G. Safonov, and G. P. Papavassilopoulos, "A Global Optimization Approach for the BMI Problem," in *Proc. IEEE Conf. on Decision and Control*, pp. 2009–2014, Dec. 1994.

[7] P. Gahinet and A. Nemirovsky, *LMILab: A Package for Manipulating and Solving LMIs.* INRIA, 1993.

[8] L. Vandenberghe and S. Boyd, SP*: Software for Semidefinite Programming. User's Guide, Beta Version.* K.U. Leuven and Stanford University, Oct. 1994.

[9] L. El Ghaoui, F. Delebecque, and R. Nikoukhah, LMITOOL*: A User-Friendly Interface for LMI Optimization, Beta Version.* ENSTA, Feb. 1995.

[10] S.-P. Wu and S. Boyd, SDPSOL: *A Parser/Solver for Semidefinite Programming and Determinant Maximization Problems with Matrix Structure. User's Guide, Beta Version.* Stanford University, June 1996.

[11] L. El Ghaoui and V. Balakrishnan, "Synthesis of Fixed-structure Controllers via Numerical Optimization," in *Proc. IEEE Conf. on Decision and Control*, pp. 2678–2683, Dec. 1994.

[12] L. El Ghaoui and J. P. Folcher, "Multiobjective Robust Control of LTI Control Design for Systems with Unstructured Perturbations," *Syst. Control Letters*, vol. 28, pp. 23–30, June 1996.

[13] L. El Ghaoui and J. P. Folcher, "Multiobjective Robust Control of LTI Systems Subject to Structured Perturbations," in *Int. Federation of Automatic Control*, pp. 179–184, June 1996.

[14] L. El Ghaoui, F. Oustry, and M. AitRami, "A Cone Complementarity Linearization Algorithm for Static Ouput Feedback and Related Problems," *IEEE Trans. Aut. Control*, 1995. *To appear.*

[15] D. Banjerdpongchai and J. P. How, "Parametric Robust $\mathcal{H}_2$ Control Design Using LMI Synthesis," in *The 1996 AIAA Guidance, Navigation, and Control Conference*, AIAA-96-3733, July 1996.

[16] D. Banjerdpongchai and J. P. How, "Parametric Robust $\mathcal{H}_2$ Control Design with Generalized Multipliers via LMI Synthesis," in *Proc. IEEE Conf. on Decision and Control*, pp. 265–270, Dec. 1996.

[17] W. Haddad and D. Bernstein, "Parameter Dependent Lyapunov Functions, Constant Real Parameter Uncertainty, and the Popov Criterion in Robust Analysis and Synthesis," in *Proc. IEEE Conf. on Decision and Control*, pp. 2274–2279, 2632–2633, Dec. 1991.

[18] J. P. How, *Robust Control Design with Real Parameter Uncertainty using Absolute Stability Theory.* PhD thesis, Massachusetts Institute of Technology, Cambridge, MA 02139, Feb. 1993.

[19] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15 of *Studies in Applied Mathematics.* Philadelphia, PA: SIAM, June 1994.

[20] A. Packard, K. Zhou, P. Pandey, and G. Becker, "A Collection of Robust Control Problems Leading to LMI's," in *Proc. IEEE Conf. on Decision and Control*, pp. 1245–1250, 1991.

[21] D. Banjerdpongchai and J. P. How, "LMI Synthesis of Parametric Robust $\mathcal{H}_\infty$ Controllers," in *Proc. American Control Conf.*, pp. 493–498, June 1997.

[22] A. G. Sparks and D. S. Bernstein, "Real Structured Singular Value Synthesis Using the Scaled Popov Criterion," *AIAA J. of Guidance, Control, and Dynamics*, vol. 18, no. 6, pp. 1244–1252, 1995.