# Synthesis of Piecewise-Affine Controllers for Stabilization of Nonlinear Systems

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Abstract— This paper presents conditions under which the feedback interconnection of a nonlinear plant and a piecewise-affine controller is stable. The piecewise-affine controller is designed for a piecewise-affine approximation of the nonlinear dynamics. The design is based on previous work of the authors. The paper also shows how to use differential inclusions to design piecewise-affine controllers for stabilization of nonlinear systems. A numerical example is provided to illustrate the procedure.

## I. Introduction

Most physical systems encountered in engineering applications are inherently nonlinear. Thus, control of nonlinear systems is a subject of active research and increasing interest. However, most controller design techniques for nonlinear systems are not systematic and/or apply only to very specific cases. Current systematic approaches to design controllers for nonlinear systems can be divided into two main categories: Lyapunov-based design and differential geometric design. A typical differential geometric approach is feedback linearization. The basic idea of this technique is to design a control law that cancels the nonlinearities of the plant and yields a closed-loop system with linear dynamics. However, the technique is not robust to uncertainties in the plant parameters, can yield to uncontrolled dynamics called zero dynamics and can only be applied to systems verifying certain vector field relations [1]. On the other hand, the primary disadvantage of Lyapunov-based techniques is the selection of the Lyapunov function, which is not systematic and far from obvious in most cases. Based on the recent developments in piecewise-affine controller design techniques [2], [3], this paper presents a new systematic Lyapunov-based design approach for nonlinear systems: design a piecewise-affine controller for a piecewise-affine approximation of the nonlinear plant and show that this controller yields stability of the original nonlinear system. The advantages of this new technique are: i) it is systematic because it searches for a parameterized control Lyapunov function with a fixed structure, ii) it can be cast as an optimization problem yielding computable controller parameters and iii) it delivers not only a controller but also a Lyapunov function that proves stability of the closed-loop system. The paper starts with a review of previous work of the authors on state feedback controller synthesis for piecewise-affine systems. Then, the main result on the stability of the feedback interconnection of a piecewise-affine state feedback controller with the original nonlinear plant is presented in Section III. The piecewise-affine controller is designed for a piecewise-affine approximation of the nonlinear system. Section III then establishes conditions on the approximation error under which it is possible to guarantee that the piecewise-affine controller designed using the techniques in [3] also stabilizes the original nonlinear plant. Section IV uses differential inclusions to guarantee a priori that the designed controller will stabilize the original nonlinear system. This is followed by a numerical example and the conclusions

## II. Controller Synthesis

The piecewise-affine control design concept for nonlinear systems is depicted in figure 1. It is assumed that the control objective is to stabilize the system to the desired closed-loop equilibrium point $x_{cl}$. Given a general nonlinear system, there are three main steps in the control design process: computing a piecewise-affine (PWA) approximation of the dynamics, designing a PWA controller for this set of dynamics, and proving that this controller stabilizes the original nonlinear plant. An algorithm for computing a piecewise-affine approximation of a class of nonlinear dynamics given a grid in the domain of the nonlinearity is presented in [4]. This section reviews the design of a state feedback piecewise-affine controller for a piecewise-affine plant as the solution to an optimization problem. Section III then presents conditions under which this controller stabilizes the original nonlinear plant.

### A. System Description

It is assumed that a piecewise-affine system and a corresponding partition of the state space with polytopic cells $\mathcal{R}_i,\ i \in \mathcal{I} = \{1, \ldots, M\}$ are given (see [4] for generating such a partition). Following [6], [5], each cell is constructed as the intersection of a finite number ($p_i$) of half spaces

$$\mathcal{R}_i = \{x \,|\, H_i^T x - g_i < 0\}, \tag{1}$$

where $H_i = [h_{i1}\ \ h_{i2} \ldots h_{ip_i}]$, $g_i = [g_{i1}\ \ g_{i2} \ldots g_{ip_i}]^T$. The sets $\mathcal{R}_i$ partition a subset of the state space $\mathcal{X} \subset I\!\!R^n$ such that $\cup_{i=1}^M \overline{\mathcal{R}}_i = \mathcal{X}$, $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$, $i \neq j$, where $\overline{\mathcal{R}}_i$ denotes the closure of $\mathcal{R}_i$. Within each cell the dynamics are affine of the form

$$\dot{x}(t) \quad = \quad A_i x(t) + b_i + B_i u(t), \tag{2}$$

where $x(t) \in I\!\!R^n$ and $u(t) \in I\!\!R^m$. Each polytopic cell has a finite number of facets and vertices. Any two cells sharing a common facet will be called level-1 neighboring cells. Let $\mathcal{N}_i = \{$level-1 neighboring cells of $\mathcal{R}_i\}$. It is also assumed that vectors $c_{ij} \in I\!\!R^n$ and scalars $d_{ij}$ exist such that the facet boundary between cells $\mathcal{R}_i$ and $\mathcal{R}_j$ is contained in the hyperplane described by $\{x \in I\!\!R^n \,|\, c_{ij}^T x - d_{ij} = 0\}$, for $i = 1, \ldots, M,\ j \in \mathcal{N}_i$. A parametric description of the boundaries can then be obtained as [5]

$$\overline{\mathcal{R}_i} \cap \overline{\mathcal{R}_j} \subseteq \{l_{ij} + F_{ij} s \,|\, s \in I\!\!R^{n-1}\} \tag{3}$$

for $i = 1, \ldots, M,\ j \in \mathcal{N}_i$, where $F_{ij} \in I\!\!R^{n \times (n-1)}$ (full rank) is the matrix whose columns span the null space of $c_{ij}$, and $l_{ij} \in I\!\!R^n$ is given by $l_{ij} = c_{ij} \left(c_{ij}^T c_{ij}\right)^{-1} d_{ij}$. For system (2), we adopt the definition of trajectories or solutions presented in [6].

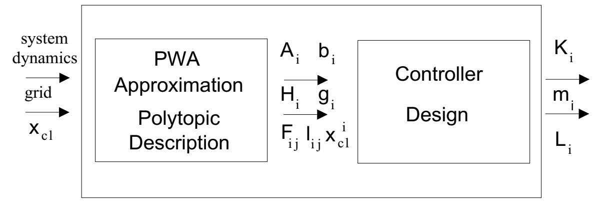


Fig. 1. Controller Design Concept

### B. Piecewise-Affine Controller Synthesis

Following previous analysis of PWA systems [5], [6], consider the piecewise-quadratic Lyapunov function continuous at the boundaries and defined in $\cup_{i=1}^M \mathcal{R}_i$ by the expression

$$\begin{aligned} V(x) &= \sum_{i=1}^{M} \beta_i(x) V_i(x), \\ V_i(x) &= \left(x^T P_i x + 2 q_i^T x + r_i\right), \end{aligned} \tag{4}$$

where $P_i = P_i^T \in I\!\!R^{(n \times n)}$, $q_i \in I\!\!R^n$, $r_i \in I\!\!R$ and

$$\beta_i(x) = \begin{cases} 1, & x \in \mathcal{R}_i \\ 0, & x \in \mathcal{R}_j,\ j \neq i \end{cases}, \tag{5}$$

for $i = 1, \ldots, M$. The expression for the candidate Lyapunov function in each region can be recast as

$$V_i(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} P_i & q_i \\ q_i^T & r_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \overline{x}^T\ \overline{P_i}\ \overline{x}. \tag{6}$$

Using the boundary description (3), continuity of the candidate Lyapunov function across the boundaries is enforced for each region $\mathcal{R}_i$ and for $j \in \mathcal{N}_i$ by [5]

$$\begin{aligned} F_{ij}^T (P_i - P_j) F_{ij} &= 0, \\ F_{ij}^T (P_i - P_j) l_{ij} + F_{ij}^T (q_i - q_j) &= 0, \\ l_{ij}^T (P_i - P_j) l_{ij} + 2(q_i - q_j)^T l_{ij} + (r_i - r_j) &= 0. \end{aligned} \tag{7}$$

Let $\alpha_i$ be the desired decay rate for this Lyapunov function in each region $\mathcal{R}_i$. Then, defining a performance criterion as $\mathcal{J} = \min_{i=1 \ldots M} \alpha_i$, the state feedback control design problem is to find from the class of control signals parameterized in the form $u = K_i x + m_i$ in each region $\mathcal{R}_i$, the one that maximizes the performance $\mathcal{J}$. The closed-loop state equations in each region $\mathcal{R}_i$ are

$$\dot{x} = (A_i + B_i K_i)\, x + (b_i + B_i m_i) \equiv \bar{A}_i x + \bar{b}_i. \tag{8}$$

The matrix $\bar{A}_i$ will be designed to be invertible and, therefore, each polytopic region will have a single equilibrium point. Setting $x_{cl}^i$ to be the closed-loop equilibrium point for region $\mathcal{R}_i$ yields the constraint

$$(A_i + B_i K_i)\, x_{cl}^i + (b_i + B_i m_i) = 0. \tag{9}$$

The function $V(x)$ in (4) will be a Lyapunov function with a decay rate of $\alpha_i$ for region $\mathcal{R}_i$ if, for fixed $\epsilon \geq 0$,

$$x \in \mathcal{R}_i \Rightarrow \begin{cases} V_i(x) > \epsilon\|x - x_{cl}\|_2, \\ \dfrac{d}{dt}V_i(x) < -\alpha_i V_i(x). \end{cases} \quad (10)$$

where $x_{cl}$ is the desired closed-loop equilibrium point of the system.

Remark 1: Note that because $V(x) > 0$ (defined in $I\!R^n$) is continuous, the fact that $V$ is piecewise-quadratic also implies that $V$ is radially unbounded, i.e., $V(x) \to +\infty$ as $\|x\| \to \infty$, provided $P_i > 0$, $i = 1, \ldots, M$. □

Using the polytopic description of the cells (1) and the $\mathcal{S}-$procedure [7], it can be shown that sufficient conditions for stability with a guaranteed decay rate of $\alpha_i$ for each region $\mathcal{R}_i$ are the existence of $P_i = P_i^T > 0$, $q_i$, $r_i$, and matrices $Z_i$ and $\Lambda_i$ with nonnegative entries satisfying

$$\begin{bmatrix} P_i - \epsilon I_n - \bar{H}_i^T Z_i \bar{H}_i & \left(q_i + \epsilon x_{cl} + \bar{H}_i^T Z_i \bar{g}_i\right) \\ (\cdot)^T & r_i - \epsilon x_{cl}^T x_{cl} - \bar{g}_i^T Z_i \bar{g}_i \end{bmatrix} > 0, \quad (11)$$

$$\begin{bmatrix} \bar{A}_i^T P_i + P_i \bar{A}_i + \bar{H}_i^T \Lambda_i \bar{H}_i & P_i \bar{b}_i + \bar{A}_i^T q_i - \bar{H}_i^T \Lambda_i \bar{g}_i \\ +\alpha_i P_i & +\alpha_i q_i \\ (\cdot)^T & 2\bar{b}_i^T q_i + \bar{g}_i^T \Lambda_i \bar{g}_i + \alpha_i r_i \end{bmatrix} < 0 \quad (12)$$

where $\bar{H}_i = [0 \ h_{i1} \ h_{i2} \ldots h_{ip_i}]^T$, $I_n$ is the identity matrix of dimension $n$, and $\bar{g}_i = [1 \ g_{i1} \ g_{i2} \ldots g_{ip_i}]^T$.

Note that conditions (11) and (12) are only concerned with the behavior of the system in the interior of the polytopic regions. To guarantee convergence of the trajectories to the closed-loop equilibrium point, it must also be ensured that the trajectories do not stay at a switching boundary for any time interval with positive length. Equivalently, additional constraints are required to ensure that sliding modes [8] are not generated at the boundaries between polytopic regions. Reference [3] shows that constraints for avoidance of sliding modes at the boundaries can be formulated as

$$\begin{aligned} c_{ij}^T \left(A_i + B_i K_i - A_j - B_j K_j\right) F_{ij} &= 0, \\ c_{ij}^T \left[\left(A_i + B_i K_i - A_j - B_j K_j\right) l_{ij}\right] + \\ c_{ij}^T \left[b_i + B_i m_i - b_j - B_j m_j\right] &= 0, \quad (13) \end{aligned}$$

for $i = 1, \ldots, M$ and $j \in \mathcal{N}_i$.

Definition 2.1: The state feedback synthesis optimization problem is

$$\begin{aligned} \max \quad & \min_i \alpha_i \\ s.t. \quad & (9), \quad (7), \quad (11), \quad (12), \quad (13) \\ & Z_i \succ 0, \ \Lambda_i \succ 0, \ \alpha_i > l_0 \geq 0, \\ & -l_1 \prec K_i \prec l_1, \ -l_2 \prec m_i \prec l_2, \ i = 1, \ldots, M, \end{aligned}$$

where $\succ$, $\prec$ mean component-wise inequalities, $l_0$ is a scalar bound and $l_1$, $l_2$ are vector bounds. Note that the optimization variables are $x_{cl}^i$, $K_i$, $m_i$, $\alpha_i$, $P_i$, $q_i$, $r_i$, $Z_i$ and $\Lambda_i$. □

Making $x_{cl}^i$ the stationary point of the quadratic sector of (4) in $\mathcal{R}_i$ yields $q_i = -P_i x_{cl}^i$. To simplify the optimization problem, the desired closed-loop equilibrium points for each polytopic region, $x_{cl}^i$, are selected a-priori (e.g, using the optimization algorithm described in [4]). Then the products of unknowns in (12) will involve only two variables and this expression is called a bilinear matrix inequality (BMI). Because of the BMI, the optimization problem 2.1 is $\mathcal{NP}$-hard. Ref. [3] presents 3 algorithms for obtaining suboptimal solutions to this problem.

Theorem 2.1: Assume the Lyapunov function (4) is defined in $\mathcal{X} \subseteq I\!R^n$. If there is a solution to the design problem from definition 2.1, the closed-loop system is locally asymptotically stable inside any subset of the largest level set of the control Lyapunov function (4) that is contained in $\mathcal{X}$. If $\epsilon > 0$ then the convergence is exponential. If, furthermore, $\mathcal{X} = I\!R^n$ then the exponential stability is global. □

Proof: See [3].

## III. Stabilization of Nonlinear Plants

This section presents the main result on the stability of the closed-loop system formed by a piecewise-affine controller and a nonlinear plant. It is assumed that a controller has been designed for a piecewise-affine system that approximates the original nonlinear system. More specifically, we assume that a nonlinear system in the form

$$\dot{x} = f(x) + g(x)u \quad (14)$$

is given for which, for simplicity, it is assumed that the full state is accessible. We further assume that a piecewise-affine approximation of these dynamics can be computed (for example using the method described in [4]) and that a piecewise-affine state feedback control law $u = K_i x + m_i$ is designed for this PWA approximation using the technique described in Section II-B. Let $\bar{A}_i$ and $\bar{b}_i$ be defined as in (8) and define the approximation error for the closed-loop system as

$$\begin{aligned}\delta(x) &= f(x) + g(x)\left(K_i x + m_i\right) - \bar{A}_i x - \bar{b}_i \\ &\equiv \tilde{f}(x) - \bar{A}_i x - \bar{b}_i \end{aligned} \qquad (15)$$

Under these assumptions, the following result can be derived.

Theorem 3.1: Assume the Lyapunov function (4) is defined in $\mathcal{X} \subseteq I\!R^n$ and define the condition number $\chi\left(\overline{P}_i\right) = \sigma_{\max}\left(\overline{P}_i\right)/\sigma_{\min}\left(\overline{P}_i\right)$ for $\overline{P}_i$ as described in (6). If there is a solution to the design problem from Definition 2.1 and if

$$\|\delta(x)\|_2 < \frac{\alpha_i\|\bar{x}\|_2}{2}\chi^{-1}\left(\overline{P}_i\right), \qquad x \in \mathcal{R}_i,$$

where $\bar{x} = [x^T \ 1]^T$, then the closed-loop system formed by the piecewise-affine state feedback controller $u = K_i x + m_i$ and the original nonlinear plant dynamics (14) is locally asymptotically stable inside any subset of the largest level set of the control Lyapunov function (4) that is fully contained in $\mathcal{X}$. If, furthermore $\mathcal{X} = I\!R^n$ then the asymptotic stability is global. □

Proof. See [9].

This is a very interesting result that says that the condition number of the $\overline{P}_i$ matrix that describes the Lyapunov function in region $\mathcal{R}_i$ should be smaller for higher modeling errors (as measured by $\|\delta(x)\|_2$). This can be interpreted as a theoretical justification for trying to minimize the maximum condition number of $\overline{P}_i,\ i = 1, \ldots, M$, as has been suggested in Ref. [3].

Note that the result in Theorem 3.1 is an a-posteriori check for stability because it concerns the state feedback controllers already designed for the piecewise-affine approximation of the nonlinear dynamics solving the optimization problem from Definition 2.1. However, the controllers can be designed so that they are guaranteed a-priori to stabilize the original nonlinear system using the method described in Section IV or using similar conditions on the norm of the approximation error as the ones suggested for stability analysis in [6].

## IV. Control of Nonlinear Systems

To synthesize piecewise-affine controllers that are guaranteed a-priori to exponentially stabilize the original nonlinear system, robustness tools such as differential inclusions [7] must be used. Differential inclusions are associated with the idea of replacing a nonlinear system by a time-varying linear system, which has been called global linearization. Differential inclusions enable us to embed the nonlinear system in the piecewise-affine framework by using time-varying system matrices and representing the nonlinear dynamics by

$$\dot{x}(t) = A_i(t)x(t) + b_i(t) + B_i(t)u(t), \ x(t) \in \mathcal{R}_i. \quad (16)$$

The following considers the case where the dynamics within each cell can be written as a convex combination of a finite set of affine dynamics, which is called a polytopic differential inclusion [7]. In other words, we assume that for every $t$ and for $x(t) \in \mathcal{R}_i$, there exist an index set $\mathcal{I}_i$, scalars $\lambda_p(t) \geq 0$, $p \in \mathcal{I}_i$, $\sum_{p\in\mathcal{I}_i}\lambda_p(t) = 1$ and matrices $(A_p, B_p, b_p)$ such that the nonlinear dynamics can be written as

$$\dot{x}(t) = \sum_{p\in\mathcal{I}_i}\lambda_p(t)\left(A_p x(t) + b_p + B_p u(t)\right), \ x(t) \in \mathcal{R}_i, \quad (17)$$

or, equivalently, $\dot{x} \in \mathbf{Co}_{p\in\mathcal{I}_i}\left\{A_p x + B_p u + b_p\right\}$, where the operator **Co** denotes the convex hull. With $u = K_i x + m_i$ in each region $\mathcal{R}_i$, the control Lyapunov function now needs to be decreasing for all possible solutions of the piecewise-affine diffferential inclusion (17). To that end, the Lyapunov function must be decreasing for every affine dynamics [7], [6]

$$\dot{x} = \left(A_p + B_p K_i\right)x + \left(b_p + B_p m_i\right) \qquad (18)$$

that define the inclusion in each cell. This condition can be written as $S < 0$ where $S$ is given by

$$\begin{bmatrix} \bar{A}_{pi}^T P_i + P_i\bar{A}_{pi} + \bar{H}_i^T\Lambda_i\bar{H}_i & P_i\bar{b}_{pi} + \bar{A}_{pi}^T q_i - \bar{H}_i^T\Lambda_i\bar{g}_i \\ +\alpha_i P_i & +\alpha_i q_i \\ (\cdot)^T & 2\bar{b}_{pi}^T q_i + \bar{g}_i^T\Lambda_i\bar{g}_i + \alpha_i r_i \end{bmatrix} \quad (19)$$

where $\bar{H}_i = [0 \ \ h_{i1} \ \ h_{i2}\ldots h_{ip_i}]^T$, $\bar{g}_i = [1 \ \ g_{i1} \ \ g_{i2}\ldots g_{ip_i}]^T$, $\bar{A}_{pi} = A_p + B_p K_i$ and $\bar{b}_{pi} = b_p + B_p m_i$. Note that when the original system is actually piecewise-affine then $\mathcal{I}_i = \{i\}$, which implies $p = i$ in (19) and (19) reduces to (12). A given point $x_{cl}^i$ can be made the desired closed-loop equilibrium point for the system from region $\mathcal{R}_i$ by making it the closed-loop equilibrium point of all affine dynamics in region $\mathcal{R}_i$, yielding the constraints

$$\bar{A}_{pi}x_{cl}^i + \bar{b}_{pi} = 0, \qquad (20)$$

for all $p \in \mathcal{I}_i$. To avoid being overly restrictive, the differential inclusion should use index sets $\mathcal{I}_i$

of the smallest possible cardinality. Defining the sliding surface between regions $\mathcal{R}_i$ and $\mathcal{R}_j$ as $\{x \in I\!R^n \,|\, \sigma_{ij} \equiv c_{ij}^T x - d_{ij} = 0\}$, then $\dot{\sigma}_{ij}$ must be continuous at the boundary described by (3) to avoid sliding modes, which yields for each region $\mathcal{R}_i$ the constraints $c_{ij}^T \left(\bar{A}_{pi}\left(F_{ij}s + l_{ij}\right) + \bar{b}_{pi}\right) = c_{ij}^T \left(\bar{A}_{qj}\left(F_{ij}s + l_{ij}\right) + \bar{b}_{qj}\right), \ \forall s \in I\!R^{n-1}, \ j \in \mathcal{N}_i$, $p \in \mathcal{I}_i$ and $q \in \mathcal{I}_j$. This can be rewritten as

$$\begin{aligned} c_{ij}^T \left(\bar{A}_{pi} - \bar{A}_{qj}\right) F_{ij} &= 0, \\ c_{ij}^T \left[\left(\bar{A}_{pi} - \bar{A}_{qj}\right) l_{ij} + \bar{b}_{pi} - \bar{b}_{qj}\right] &= 0, \end{aligned} \quad (21)$$

for $i = 1, \ldots, M$, $j \in \mathcal{N}_i$, $p \in \mathcal{I}_i$ and $q \in \mathcal{I}_j$.

Definition 4.1: The state feedback synthesis optimization problem is

$$\begin{aligned} \max \quad & \min_i \alpha_i \\ s.t. \quad & (20), \quad (7), \quad (11), \quad (19), \quad (21) \\ & Z_i \succ 0, \ \Lambda_i \succ 0, \ \alpha_i > l_0 \geq 0, \\ & -l_1 \prec K_i \prec l_1, \ -l_2 \prec m_i \prec l_2, \end{aligned}$$

where $i = 1, \ldots, M, \ j \in \mathcal{N}_i, \ p \in \mathcal{I}_i, \ t \in \mathcal{I}_j$, $l_0$ is a scalar bound and $l_1$, $l_2$ are vector bounds. □

We can now state the following Corollary of Theorem 2.1.

Corollary 4.1: Assume the Lyapunov function (4) is defined in $\mathcal{X} \subseteq I\!R^n$. If there is a solution to the design problem from Definition 4.1, then the closed-loop system is locally asymptotically stable inside any subset of the largest level set of the control Lyapunov function (4) that is fully contained in $\mathcal{X}$. If $\epsilon > 0$ then the convergence is exponential. If furthermore $\mathcal{X} = I\!R^n$, exponential stability is global.

Proof: See Ref. [9].

Note that this result is more powerful than the one in Theorem 3.1 because it is an a-priori result that guarantees that the resulting controller will exponentially stabilize the original nonlinear system. Similarly to the standard results in differential inclusions, Theorem 4.1 has the drawback of being potentially conservative if the convex hull in which the nonlinear system is embedded does not have a "minimal" size (in some sense). Ref. [7] discusses several methods for reducing the conservatism in modeling systems using differential inclusions. Furthermore, note that if a globally quadratic Lyapunov function is used, then $x_{cl}^i = x_{cl}, \ i = 1, \ldots, M$, and stability is proved for any switching [10], which enables one to remove the conditions (20) and (21).

## V. Example

Consider a temperature exchanger system whose temperature (in degrees Celsius) is bounded, belonging to the interval $[-2, 2]$, and obeys the first-order nonlinear differential equation $\dot{T} = 0.5\left(1 - T^2\right) + u$. A state feedback controller was designed for the PWA approximation of the system described by $\dot{T} = 1 - |T| + u$. Setting $x = T$ generates the polytopic regions $\mathcal{R}_1 = \{x \in I\!R \mid -2 < x < 0\}$ and $\mathcal{R}_2 = \{x \in I\!R \mid 0 < x < 2\}$. The objective is to stabilize the open-loop unstable equilibrium point of region $\mathcal{R}_1$ at $x = -1$, with the choice of equilibrium points $x_{cl}^1 = x_{cl}^2 = -1$. Using the method described in section II-B with $l_0 = 0, \ l_1 = l_2 = 10, \ \epsilon = 10^{-3}$, after two iterations of the $V$-$K$ algorithm [3] the results are $\alpha_1 = \alpha_2 = 18, \ \chi\left(\overline{P}_1\right) = \chi\left(\overline{P}_2\right) = 2.618$. Since $\|\bar{x}\| = \sqrt{(1 + \|x\|^2)} \geq 1$, invoking Theorem 3.1, we find that the controller stabilizes the original nonlinear system if

$$\|\delta\| = \|0.5\left(1 - x^2\right) - 1 + |x|\| < 3.44$$

Comparing the parabola with the upper line in figure 2 it is seen that indeed $\|\delta\| \leq 0.5 < 3.44$. The resulting controller parameters are $K_1 = -10, \ K_2 = -8, \ m_1 = -10, \ m_2 = -10$. If the nonlinearity is bounded by the straight lines shown in figure 2, then the method of differential inclusions searching for a globally quadratic Lyapunov function, yields after one iteration $P_1 = P_2 = 1, \ r_1 = r_2 = 1.6, \ K_1 = -2.41, \ K_2 = -0.41, \ m_1 = -2.41, \ m_2 = -2.41$ (the performance of the controller does not improve after the first iteration). Invoking Corollary 4.1, this controller stabilizes the nonlinear system. A comparison of the controllers obtained using the two different methods is shown in the simulations presented in figure 3 for the initial condition $x_0 = 1$ (inside $\mathcal{R}_2$). It is clear from the plot that, as expected, the method of differential inclusions yielded a more conservative result (slower controller).

## VI. Conclusions

This paper presented conditions under which the feedback interconnection of a nonlinear plant and a piecewise-affine controller (designed for a piecewise-affine approximation of the nonlinear dynamics) is stable. The paper has also shown how to use the concept of differential inclusions to design piecewise-affine controllers for stabilization of nonlinear systems. Together with previous work of the authors

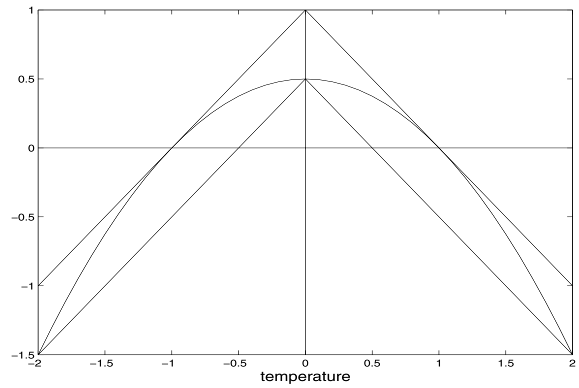


Fig. 2. Nonlinearity and its PWA approximations.

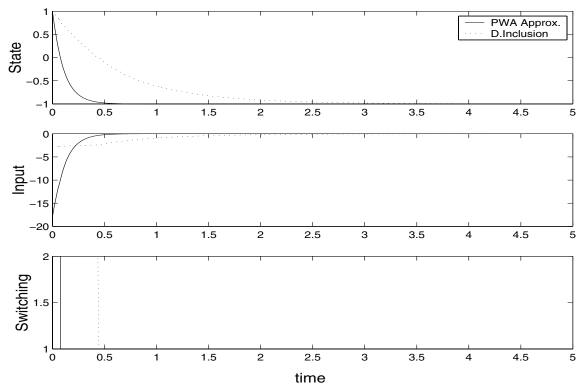


Fig. 3. Simulation Results.

[3], this paper enables a fully automated synthesis tool for a wide class of nonlinear systems.

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