Parametric Robust $\mathcal{H}_\infty$ Controller Synthesis: Comparison and Convergence Analysis[1]

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Abstract

Recent papers have demonstrated the effectiveness of our iterative algorithm using linear matrix inequalities (LMI's) on several parametric robust $\mathcal{H}_\infty$ control designs. This paper presents two additional important components to the discussion on the behavior of the new LMI-based iterative algorithm: a comparison study between the LMI synthesis technique and the existing iterative approaches of the complex and mixed μ/K_m synthesis, and a convergence analysis of this new algorithm. The results indicate that the Popov $\mathcal{H}_\infty$ controller synthesis provides a viable alternative for designing real parametric robust controllers and exhibits properties similar to the D–K and D,G–K iteration of the complex and mixed μ/K_m synthesis. The key potential advantage of using the LMI approach is the elimination of the curve-fitting for the D and G scaling functions.

1 Introduction

An LMI-based algorithm has recently been developed to solve for parametric robust controllers such that the stability of the Lur'e system is guaranteed, and its performance bound ($\mathcal{H}_2$ or $\mathcal{H}_\infty$) is minimized [1, 2]. In particular, we have shown that the design formulation results in Bilinear Matrix Inequalities (BMI's) because there are product terms involving compensator parameters and the Popov multiplier parameters. Our solution approach for these nonconvex optimization problems is based on an iterative procedure. The proposed algorithm, called the *V–K iteration* [3, 4], alternates between three different LMI problems and is repeated until the stopping criterion is satisfied. Previous analysis has verified that, for parametric robust $\mathcal{H}_2$, this solution procedure converges to an optimal solution [5].

In this paper, we add two important elements to the discussion on the behavior of the new LMI-based iterative algorithm when used to design parametric robust $\mathcal{H}_\infty$ controllers. First, we compare controllers designed by the LMI synthesis technique with ones calculated by the existing iterative approaches of the complex and mixed μ/K_m synthesis [6, 7]. Second, we analyze the convergence of the new iterative algorithm to verify that it efficiently converges to the optimal solution. These results indicate that the new iterative algorithm exhibits properties similar to the D–K and D,G–K iteration of the complex and mixed μ/K_m synthesis and offers key advantages over the existing design tools.

2 Comparison with the D,G–K Iteration

The first objective is to compare controllers designed by the LMI-based iterative algorithm [2] with ones designed using the existing design tools [6, 7]. A benchmark problem is a cantilevered Bernoulli Euler beam with unit length and mass density, and stiffness scaled so that the fundamental frequency is 1 rad/sec [8, 2]. Note that this problem was chosen so that the realness of the parametric uncertainty in the design problem is accentuated. To proceed, a frequency dependent weighting function, $W(s)$, was placed on the disturbance to the performance loop. For this study, we choose $W(s) = 10(s + 40)/(s + 400)$.

A controller was designed for $\pm 1\%$ variation of the third modal frequency using the Popov $\mathcal{H}_\infty$ controller synthesis (PCS) technique. The stopping criterion is 1% absolute and relative accuracy of the $\mathcal{H}_\infty$ cost overbound [2]. The complex and mixed μ controllers are designed for the same frequency variation. Note that for the conventional μ/K_m synthesis framework, the uncertainty blocks are free to vary as well as the performance block. However, to make the comparison of these controllers fair and consistent, we will fix the size of the uncertainty and then compute the lowest upper bound of the worst case $\mathcal{H}_\infty$ performance.

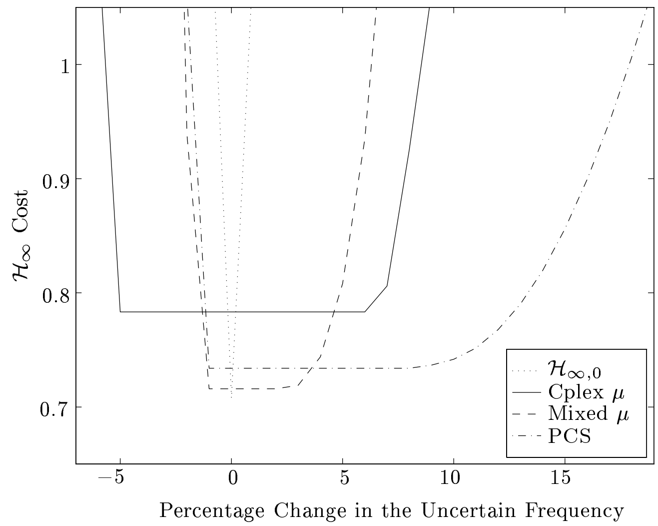

Fig. 1: Robust performance comparison for the complex μ, mixed μ, and Popov $\mathcal{H}_\infty$ controller synthesis with $\pm 1\%$ guaranteed stability bounds.

A representative post-analysis of the four control design techniques is shown in Figure 1. The figure clearly shows that all robust controllers achieve the desired stability bounds ($\pm 1\%$). Moreover, the actual performance for all controllers is quite flat in the region of guaranteed robustness. Controllers were also designed to provide guarantees of $\pm 2\%$ and $\pm 3\%$ in the modal frequency, and the post-analysis yielded results similar to those shown in Figure 1. However, due to space limitations, these plots are not included. Figure 1 shows that the complex μ controller yields conservative performance (*i.e.*, the $\mathcal{H}_\infty$ cost is higher than that of the mixed μ and Popov $\mathcal{H}_\infty$ controller designs) when analyzed with the real uncertainty. This result is expected because, by covering the real uncertainty with a complex one, complex μ admits a much larger class of allowable uncertainties. The figure clearly illustrates a significant reduction in the $\mathcal{H}_\infty$ cost when we use the mixed μ and Popov $\mathcal{H}_\infty$ controllers techniques.

Note that in the Popov $\mathcal{H}_\infty$ controller synthesis (PCS), both the analysis and synthesis steps in the V–K iteration are formulated using a state space approach. On the other hand, the mixed μ synthesis is based on a structured singular value calculation in the frequency domain with curve-fitting of the scaling functions D and G, and then a separate $\mathcal{H}_\infty$ controller calculation. Curve fitting errors of D and G could lead to conservatism in the mixed μ synthesis, as discussed in Ref. [9] for the complex μ synthesis.

Table 1 summarizes the control order and the key points for the achieved robust stability bounds and performance for Figure 1. For

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Table 1: Achieved robust stability and $\mathcal{H}_\infty$ performance for complex μ, mixed μ, and Popov $\mathcal{H}_\infty$ controller synthesis with $\pm 1\%$ guaranteed robust bound.

Type of Control Design	Control Order	% Change of Nominal $\mathcal{H}_\infty$ Cost	Lower Stability Bound %	Upper Stability Bound %
Complex μ	11	10.68	-6	27
Mixed μ	13	1.16	-4	12
PCS	9	3.71	-4	63

the mixed μ synthesis and PCS, the achieved lower stability bounds are very close to the guaranteed robustness bounds and smaller than the achieved bounds using the complex μ design. However, the achieved upper robustness bound of PCS is significantly larger than that of the complex and mixed μ synthesis.

This robust stability and $\mathcal{H}_\infty$ performance analysis provides insights into the issue of conservatism for the design techniques. Although the complex μ controllers are appropriate for systems with structured complex uncertainty, it obviously becomes conservative for the case of real parametric uncertainty. These results confirm that the mixed μ synthesis and the Popov $\mathcal{H}_\infty$ controller synthesis provide a means to capture the phase information of the real uncertainty which results in a reduction of the conservatism in the control designs. Thus the Popov $\mathcal{H}_\infty$ controller synthesis potentially provides an alternative approach to design controllers for systems subject to real parametric uncertainty. The key potential advantage of using the LMI approach is the elimination of the curve-fitting for the scaling functions.

3 Convergence Analysis of the V–K Iteration

The second objective is to demonstrate the convergence of the V–K iteration for designing the PCS in the previous section. The algorithm yields a large reduction of the performance bound in the first few iterations, but only a small reduction in the subsequent iterations. With the stopping criterion at 1% of the absolute and relative accuracy, the algorithm terminates after 3 iterations. A total of 18 iterations are required with a stopping accuracy of 0.01%, and when the accuracy is decreased to 0.001%, the algorithm continues until the 60^{th} iteration.

As discussed in [5], we compute the optimal solution by a simple exhaustive search. The mesh plots shown in Figure 2 illustrates the worst case $\mathcal{H}_\infty$ cost overbounds for various multiplier parameters (shown by the surface mesh and contours of constant upper bound). A very large coarse grid was searched first. This figure displays the results on a refined grid around the only local minimum found in the entire grid space searched. The first observation from the figure is that, in stark contrast to the $\mathcal{H}_2$ results [5], the worst case $\mathcal{H}_\infty$ cost overbound does not behave like a convex function in this region of the multiplier parameters near the optimal solution.

The initial multiplier (not shown in the plot) lies far outside the region of the mesh plot. With a 1% accuracy for the stopping criterion, the algorithm yields an upper bound of the cost which is 1.77% higher than the optimal value. Thus the V–K algorithm yields a solution that is close to the optimum, and, as shown in the plot, can be improved with further iterations (the upper bound of the cost is only 0.37% above the optimal value after 18 iterations). The typical time of each iteration is approximately 2.7 minutes on Sun-Sparc 20/60, so the total design time for a tolerance of 1% is only 8.1 minutes. It is interesting to note that this rate of convergence is slower than that achieved when a similar algorithm was applied to the parametric robust $\mathcal{H}_2$ control design problem [5].

Note that in this new algorithm there are shared variables between each phase of the iteration, specifically, the quadratic term of the Lyapunov function [1, 2]. This characteristic is not present in the D,G–K iteration technique which consists of a D and G phase (the robust analysis) that is entirely separate from the K phase (the $\mathcal{H}_\infty$ synthesis). The shared variables in the LMI synthesis are thought

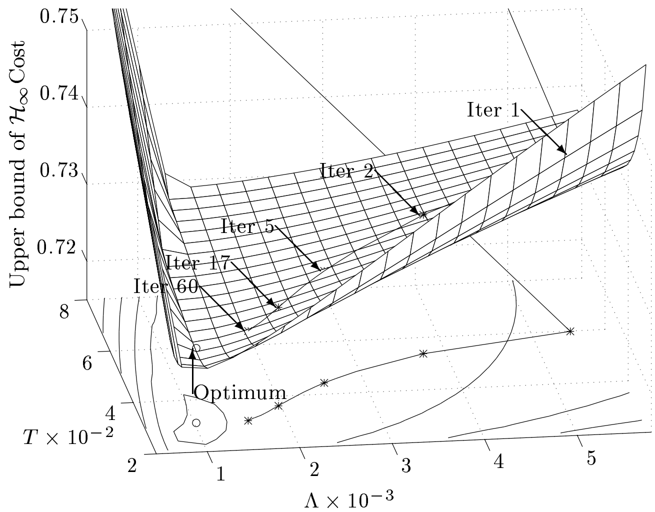

Fig. 2: Mesh plot of the $\mathcal{H}_\infty$ cost overbounds as a function of the multiplier parameters (Λ and T).

to play a key role in the convergence of this new algorithm to a local optimum, which represents a significant improvement over the performance of the original V–K iteration discussed in Refs [3, 4].

4 Conclusions

This paper presents two additional important results on the real parametric robust $\mathcal{H}_\infty$ controller synthesis. First, the robust performance of the LMI-based iterative algorithm is shown to compare very favorably with the complex and mixed μ synthesis algorithms. The numerical results indicate that the Popov $\mathcal{H}_\infty$ controller synthesis provides a viable alternative for designing real parametric robust controllers. Second, we verify that the V–K iteration converges close to the optimal solution. The benefits of the LMI-based algorithm are that the structure of the controller matrices is exploited and the curve-fitting in the design procedure is eliminated. However, the current implementation of the LMI design framework tends to require significantly more memory for large order systems.

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References

[1] D. Banjerdpongchai and J. P. How, "Parametric Robust $\mathcal{H}_2$ Control Design Using LMI Synthesis," in *AIAA Guidance, Navigation, and Control Conference,* AIAA-96-3733, July 1996.

[2] D. Banjerdpongchai and J. P. How, "LMI Synthesis of Parametric Robust $\mathcal{H}_\infty$ Controllers," in *Proc. American Control Conf.*, pp. 493–498, June 1997.

[3] S. Boyd, "Control system analysis and synthesis via linear matrix inequalities," in *Proc. American Control Conf.*, 1993.

[4] L. El Ghaoui and V. Balakrishnan, "Synthesis of Fixed-structure Controllers via Numerical Optimization," in *Proc. Conf. on Decision and Control*, pp. 2678–2683, Dec. 1994.

[5] D. Banjerdpongchai and J. P. How, "Convergence Analysis of a Parametric Robust $\mathcal{H}_2$ Controller Synthesis Algorithm," in *Proc. Conf. on Decision and Control*, pp. 1020–1025, Dec. 1997.

[6] R. Y. Chiang and M. G. Safanov, *Robust Control Toolbox.* The Mathworks Inc., 1992.

[7] G. J. Balas, J. C. Doyle, K. Glover, A. Packard, and R. Smith, *μ-Analysis and Synthesis Toolbox.* The Mathworks Inc., 1993.

[8] S. C. O. Grocott, J. P. How, and D. W. Miller, "Comparison of Robust Control Techniques for Uncertain Structural Systems," in *AIAA Guidance, Navigation, and Control Conference*, pp. 261–271, Aug. 1994.

[9] K. C. Goh, J. H. Ly, L. Turand, and M. G. Safonov, "μ/k_m-Synthesis via Bilinear Matrix Inequalities," in *Proc. Conf. on Decision and Control*, pp. 2032–2037, Dec. 1994.