LMI Synthesis of Parametric Robust $\mathcal{H}_\infty$ Controllers[1]

David Banjerdpongchai[2]

Durand Bldg., Room 110
Dept. of Electrical Engineering
Email: banjerd@isl.stanford.edu

Jonathan P. How

Durand Bldg., Room 277
Dept. of Aeronautics and Astronautics
Email: howjo@sun-valley.stanford.edu

Stanford University, Stanford CA 94305

Abstract

This paper presents a new algorithm for designing full order LTI controllers for systems with real parametric uncertainty. The approach is based on the robust $\mathcal{L}_2$ gain analysis of the Lur'e system using Popov analysis and multipliers. The core algorithm, previously applied to the robust $\mathcal{H}_2$ performance synthesis problem, is shown to be applicable to the robust controller design with the $\mathcal{H}_\infty$ cost. Although the performance metrics are different, we demonstrate that the same solution algorithm based on LMI synthesis leads to a very effective and efficient technique for real parametric robust $\mathcal{H}_\infty$ control design. Furthermore, it is difficult to compare robust $\mathcal{H}_2$ controllers to μ/K_m designs, but in this work we provide insights into the issue of conservatism for various robust $\mathcal{H}_\infty$ control approaches, in particular, the Popov controller synthesis, the robust $\mathcal{H}_\infty$ design, and the μ/K_m synthesis. The detailed analysis of these approaches is demonstrated on a flexible structure benchmark problem.

Keywords: Lur'e system, real parametric uncertainty; $\mathcal{L}_2$ gain; Popov controller synthesis; bilinear matrix inequality; linear matrix inequality.

1 Introduction

Robust $\mathcal{H}_\infty$ control problems for complex/real parametric uncertainty have been studied in detail since the introduction of the structured singular value, μ, by Doyle [1] and the multivariable stability margin, K_m, by Safonov [2]. It was shown that the optimally scaled singular values produce a nonconservative estimate of the structured singular value. A hybrid of the $\mathcal{H}_\infty$ control theory and the diagonal scaling techniques for the μ/K_m synthesis has proven to be effective for designing robust controllers for systems with complex uncertainty. The synthesis based on *the D–K iteration* was originally devised by Dolye [3] and Safonov [4]. One major drawback with the D–K iteration approach is that it requires curve fitting approximations after each D iteration which can significantly increase the designer's input during the controller synthesis. Due to the difficulties in curve fitting for the complex and real parametric uncertainty case, Safonov and Chiang [5] showed that the curve-fitting from the design procedure can be eliminated. The full μ/K_m synthesis problem using *Bilinear Matrix Inequalities (BMI's)* is formulated in Refs. [6, 7]. A key benefit of the BMI approach is that the compensator architecture (reduced order, decentralized control) can be included in the design framework. However, the problem size in this formulation is quite large because the structure of the closed-loop system has not been fully exploited. Moreover, the problem of how to optimally select the basis of the scaling multipliers is still unresolved.

In this paper we investigate the μ/K_m synthesis problem by applying the Popov absolute stability analysis to the Lur'e system [8]. The design objective of minimizing an upper bound of the $\mathcal{L}_2$ gain, *i.e.*, robust $\mathcal{H}_\infty$ performance for real parametric uncertain systems, naturally leads to BMI's. El Ghaoui and Balakrishnan [9] propose an iterative solution procedure for these BMI's using a two-stage optimization process, called the V–K iteration. We have already successfully applied an extension of this algorithm to parametric robust $\mathcal{H}_2$ control design problem with Popov multipliers [10] and generalized multipliers [11]. This paper presents a similar algorithm for the μ/K_m synthesis problem. A unique feature of this work is that we use the same core algorithm to solve both the parametric robust $\mathcal{H}_2$ and $\mathcal{H}_\infty$ synthesis problems.

Our problem statement is similar to the one in Refs. [6, 7], but we restrict our attention to the system subject to sector bounded nonlinear uncertainty. Moreover, we take advantage of the closed-loop system structure to eliminate some design parameters from the problem formulation using a simple algebraic technique [12]. This well-known approach significantly reduces the problem size and the number of design parameters. However, the coupling in the BMI is not completely removed when the multipliers are added to the design problem. Hence, an iterative algorithm is still required, but it is quite distinct from the D–K iteration for the μ/K_m synthesis. For example, some variables are shared between the two main stages of our iterative solution and we conjecture that this plays an important role in the efficiency and robustness of the solution approach. A slight computational advantage in our design framework is that the overbound of the robust performance can be simultaneously minimized over the design parameters. This allows us to bypass the *γ-iteration* in the previous μ/K_m design approach. Thus, while the multipliers in this paper are not as general as the ones in μ/K_m synthesis, this technique offers an alternative to the well-known D–K iteration based synthesis algorithms. Our procedure also eliminates the curve-fitting of the real structured singular value. In addition, the problem size is smaller than the BMI's of the original formulation of the μ/K_m synthesis. With further investigation, these combined benefits

[1] This research was supported by Ananda Mahidol Foundation and in part by AFOSR under F49620-95-1-0318.

[2] Author to whom all correspondence should be addressed. Tel: (415) 723-9833; Fax: (415) 723-8473.

could lead to a more robust solution algorithm for the control synthesis of mixed uncertain systems and mixed performance objectives.

2 Problem Statement

We consider an LTI system subject to sector bounded nonlinear uncertainty, *i.e.*, a Lur'e system (see Figure 1), described by

$$\begin{array}{rcl} \dot{x} & = & Ax + B_p p + B_w w + B_u u \\ q & = & C_q x + D_{qp} p + D_{qw} w + D_{qu} u \\ z & = & C_z x + D_{zp} p + D_{zw} w + D_{zu} u \\ y & = & C_y x + D_{yp} p + D_{yw} w + D_{yu} u \\ p & = & \phi(q), \end{array} \qquad (1)$$

where $x : \mathbf{R}_+ \rightarrow \mathbf{R}^n$ is the state, $u : \mathbf{R}_+ \rightarrow \mathbf{R}^{n_u}$ is the control input, $w : \mathbf{R}_+ \rightarrow \mathbf{R}^{n_w}$ is the disturbance input, $y : \mathbf{R}_+ \rightarrow \mathbf{R}^{n_y}$ is the measured output and $z : \mathbf{R}_+ \rightarrow \mathbf{R}^{n_z}$ is the performance output. $p : \mathbf{R}_+ \rightarrow \mathbf{R}^{n_p}$ are the input/output of the nonlinear uncertainty ϕ. The nonlinear perturbation ϕ is assumed to satisfy the sector bound $[0,\ 1]$, *i.e.*, $\phi \in \mathbf{\Phi}$ where $\mathbf{\Phi} := \{\phi : \mathbf{R}^{n_p} \rightarrow \mathbf{R}^{n_p},\ \phi(q) = [\phi_1(q_1), \ldots, \phi_{n_p}(q_{n_p})]^T,$ where $0 \leq \phi_i(\sigma)/\sigma \leq 1,\ \forall i = 1, \ldots, n_p\}$. As discussed in Ref. [12, page 129], a loop transformation can be used to handle the more general sector condition $\alpha_i \leq \phi_i(\sigma)/\sigma \leq \beta_i$. The description of the Lur'e system also includes an important class of uncertain systems described by $\dot{x} = (A + \Delta A)x + B_w w + B_u u$, $\Delta A \in \mathcal{U}$, where $\mathcal{U} := \{\Delta A \in \mathbf{R}^{n \times n} : \Delta A = B_p D C_q,\ D = \mathbf{diag}(\delta_1, \ldots, \delta_{n_p}),$ where $\delta_i \in [0,\ 1],\ \forall i = 1, \ldots, n_p\}$. In control theory, this is referred to as the system subject to *real parametric uncertainty* [13, 14]. This special case of the Lur'e system (1) occurs when the functions ϕ_i are linear, *i.e.*, $\phi_i(\sigma) = \delta_i \sigma$, where $\delta_i \in [0,\ 1]$, $\forall i = 1, \ldots, n_p$. To significantly simplify the analysis and synthesis, we assume D_{zp}, D_{zw}, D_{qp}, D_{qw}, and D_{qu} are identically zero.

The objective of this paper is to design a strictly proper full order LTI controller using Popov absolute stability theory for the system (1) such that the robust stability of the system is achieved and an overbound of the $\mathcal{L}_2$ gain is minimized.

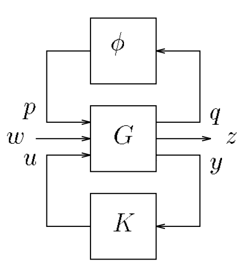

Figure 1: Elements of the robust synthesis problem

Let $U \in \mathbf{R}^{n \times p}$. $U_\perp$ is defined as an orthogonal complement of U, *i.e.*, $U^T U_\perp = 0$ and $[U\ U_\perp]$ is of maximum rank. $\mathcal{L}_2^n$ is the Hilbert space of square-integrable signals defined over $\mathbf{R}_+$ with n components, *i.e.*, $w \in \mathcal{L}_2^n$ satisfying $\int_0^\infty w^T w\, dt < \infty$. $\mathcal{L}_2^n$ is often abbreviated as $\mathcal{L}_2$. A causal n-input n-output operator $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is said to be $\mathcal{L}_2$ *stable* if there exist $\gamma \geq 0$ and β such that

$$\|Fw\|_2 \ \leq\ \gamma \|w\|_2 + \beta,\ \forall w \in \mathcal{L}_2, \qquad (2)$$

where $\|\cdot\|_2$ is defined as the $\mathcal{L}_2$ *norm*. The $\mathcal{L}_2$ *gain* of F is defined as the smallest γ such that (2) holds for some β.

2.1 Popov Robust $\mathcal{H}_\infty$ Performance Analysis

The Popov robust stability analysis is based on Lyapunov functions of the form

$$V(x) = x^T P x + 2 \sum_{i=1}^{n_p} \lambda_i \int_0^{C_{i,q} x} \phi_i(\sigma)\, d\sigma \qquad (3)$$

where $C_{i,q}$ denotes the i^{th} row of C_q. Thus the data describing the Lyapunov function are the matrix P and the scalars λ_i, $i = 1, \ldots, n_p$. We require $P > 0$ and $\lambda_i \geq 0$, which implies that $V(x) \geq x^T P x > 0$ for nonzero x. For the case when $\phi_i(\sigma) = \delta_i \sigma$, *i.e.*, linear or real parametric uncertainty, the Lyapunov function will have the form $V(x) = x^T(P + C_q^T D \Lambda C_q)x$, where $D = \mathbf{diag}(\delta_1, \ldots, \delta_{n_p})$ and $\Lambda = \mathbf{diag}(\lambda_i, \ldots, \lambda_{n_p})$. This Lyapunov function is referred to as a *parameter-dependent Lyapunov function* [13, 14]. For the nonlinear system (1), the robust performance is derived from the $\mathcal{L}_2$ gain, *i.e.*, the RMS gain. While the exact $\mathcal{L}_2$ gain of the system (1) is difficult to compute, its upper bound can be easily computed as shown in the following theorem.

Theorem 1 *([12]) If there exists a Lyapunov function of the form (3),* $\Lambda := \mathbf{diag}_{i=1}^{n_p}(\lambda_i) \geq 0$, $T := \mathbf{diag}_{i=1}^{n_p}(\tau_i) \geq 0$, *and* $\gamma^2 > 0$ *satisfying*

$$\begin{bmatrix} \begin{array}{c} A^T P + PA + \\ C_z^T C_z \end{array} & \begin{array}{c} PB_p + \\ A^T C_q^T \Lambda + C_q^T T \end{array} & PB_w \\ \begin{array}{c} B_p^T P + \\ \Lambda C_q A + T C_q \end{array} & \begin{array}{c} \Lambda C_q B_p + \\ B_p^T C_q^T \Lambda - 2T \end{array} & \Lambda C_q B_w \\ B_w^T P & B_w^T C_q^T \Lambda & -\gamma^2 I \end{bmatrix} \leq 0, \qquad (4)$$

then the upper bound on the $\mathcal{L}_2$ gain is finite and can be obtained by solving the optimization problem of minimizing γ^2 over the variables γ^2, P, Λ, and T, i.e.,

$$\begin{array}{ll} \textit{minimize} & \gamma^2 \\ \textit{subject to} & (4),\ P > 0,\ \Lambda \geq 0,\ T \geq 0. \end{array} \qquad (5)$$

Proof. *See [12, page 122].*

Note that while Ref. [12] states the convex optimization technique for analyzing the robust $\mathcal{H}_\infty$ performance, it provides no insight on how to solve the synthesis problem. In the following subsection, we will use this Popov robust $\mathcal{H}_\infty$ performance analysis as a tool to design robust compensators.

2.2 Popov Controller Synthesis

Our design goal is to find a strictly proper full order LTI controller that minimizes the upper bound of the $\mathcal{L}_2$ gain derived in the preceding subsection. The controller is of the form

$$\dot{x}_c \ =\ A_c x_c + B_c y, \quad u \ =\ C_c x_c, \qquad (6)$$

where $x_c : \mathbf{R}_+ \rightarrow \mathbf{R}^n$ is the controller state; A_c, B_c, and C_c are constant matrices of appropriate size. The closed-loop system of the Lur'e system (1) and the LTI controller (6), shown in Figure 1, is described by

$$\begin{array}{rcl} \dot{\tilde{x}} & = & \tilde{A}\tilde{x} + \tilde{B}_p p + \tilde{B}_w w \\ q & = & \tilde{C}_q \tilde{x} + \tilde{D}_{qp} p + \tilde{D}_{qw} w \\ z & = & \tilde{C}_z \tilde{x} + \tilde{D}_{zp} p + \tilde{D}_{zw} w \\ p & = & \phi(q), \end{array} \qquad (7)$$

where

$$\left[\begin{array}{c|c|c} \tilde{A} & \tilde{B}_p & \tilde{B}_w \\ \hline \tilde{C}_q & \tilde{D}_{qp} & \tilde{D}_{qw} \\ \hline \tilde{C}_z & \tilde{D}_{zp} & \tilde{D}_{zw} \end{array}\right] =$$

$$\left[\begin{array}{cc|c|c} A & B_u C_c & B_p & B_w \\ B_c C_y & A_c + B_c D_{yu} C_c & B_c D_{yp} & B_c D_{yw} \\ \hline C_q & D_{qu} C_c & D_{qp} & D_{qw} \\ \hline C_z & D_{zu} C_c & D_{zp} & D_{zw} \end{array}\right],$$

and $\tilde{x}^T = [\ x^T \quad x_c^T\ ]$. Then it is straightforward to compute the upper bound of the $\mathcal{L}_2$ gain for the closed-loop system (7). We note that (4) is equivalent to

$$\left[\begin{array}{ccc} \tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} + \tilde{C}_z^T\tilde{C}_z & \tilde{P}\tilde{B}_p + \tilde{A}^T\tilde{C}_q^T\Lambda + \tilde{C}_q^T T & \tilde{P}\tilde{B}_w \\ \tilde{B}_p^T\tilde{P} + \Lambda\tilde{C}_q\tilde{A} + T\tilde{C}_q & \Lambda\tilde{C}_q\tilde{B}_p + \tilde{B}_p^T\tilde{C}_q^T\Lambda - 2T & \Lambda\tilde{C}_q\tilde{B}_w \\ \tilde{B}_w^T\tilde{P} & \tilde{B}_w^T\tilde{C}_q^T\Lambda & -\gamma^2 I \end{array}\right] \leq 0. \quad (8)$$

In summary, the design objective is to solve the non-convex optimization problem over the parameters γ^2, $\tilde{P}$, Λ, T, A_c, B_c, and C_c.

$$\begin{array}{ll} \text{minimize} & \gamma^2 \\ \text{subject to} & (8),\ \tilde{P} > 0,\ \Lambda \geq 0,\ T \geq 0. \end{array} \quad (9)$$

3 Design Procedure

This section closely parallel the developments in Refs. [15, 10, 11]. As will be shown, controllers are developed in two main steps. Observing the structure of the compensator parameters in (8), the first step is to eliminate some controller parameters from the problem formulation (9). We then solve for the remaining variables, and use these results to construct the controllers. An iterative algorithm is required to calculate the controllers, but in the process the procedure capitalizes on the very efficient design tools that are available for solving *Linear Matrix Inequalities (LMI's)* [16, 17]. The resulting compensators are full-order, and cannot include architecture constraints. However, the solution procedure is very robust, which significantly reduces the user workload. Furthermore, this approach is easily expandable to include other sophisticated analysis tests such as analysis for systems with mixed uncertainty or linear time-invariant uncertainty.

3.1 Controller Elimination

We first note that the controller matrix A_c only appears in (8). Thus it is possible to reduce the number of variables in the problem by eliminating A_c. To proceed, we define

$$\tilde{A}_0 := \begin{bmatrix} A & B_u C_c \\ B_c C_y & B_c D_{yu} C_c \end{bmatrix}, \quad \tilde{J} := \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Then $\tilde{A}$ can be written as $\tilde{A} = \tilde{A}_0 + \tilde{J}A_c\tilde{J}^T$ and we rewrite (8) as

$$\tilde{G} + VA_c^TV^T + UA_cU^T < 0, \quad (10)$$

where $\tilde{G}$, V, and U are defined as

$$\tilde{G} = \left[\begin{array}{ccc} \tilde{A}_0^T\tilde{P} + \tilde{P}\tilde{A}_0 + \tilde{C}_z^T\tilde{C}_z & \tilde{P}\tilde{B}_p + \tilde{A}_0^T\tilde{C}_q^T\Lambda + \tilde{C}_q^T T & \tilde{P}\tilde{B}_w \\ \tilde{B}_p^T\tilde{P} + \Lambda\tilde{C}_q\tilde{A} + T\tilde{C}_q & \Lambda\tilde{C}_q\tilde{B}_p + \tilde{B}_p^T\tilde{C}_q^T\Lambda - 2T & \Lambda\tilde{C}_q\tilde{B}_w \\ B_w^T\tilde{P} & \tilde{B}_w^T\tilde{C}_q^T\Lambda & \gamma^2 \end{array}\right],$$

$$V^T := [\ \tilde{J}^T \quad 0 \quad 0\ ], \quad U^T := [\ \tilde{J}^T\tilde{P}^T \quad 0 \quad 0\ ].$$

Applying the Elimination Lemma [12, page 32], we first note that the orthogonal complements of V and U are as following.

$$V_\perp = \begin{bmatrix} \tilde{J}_\perp & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad U_\perp = \begin{bmatrix} \tilde{P}^{-1}\tilde{J}_\perp & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Then, it follows that (10) holds if and only if

$$V_\perp^T\tilde{G}V_\perp < 0, \quad U_\perp^T\tilde{G}U_\perp < 0. \quad (11)$$

To proceed, we partition $\tilde{P}$ and its inverse $\tilde{Q}$ as

$$\tilde{P} = \begin{bmatrix} P & M \\ M^T & R \end{bmatrix}, \quad \tilde{Q} = \tilde{P}^{-1} = \begin{bmatrix} Q & N \\ N^T & S \end{bmatrix}, \quad (12)$$

where P and $Q \in \mathbf{R}^{n\times n}$. N is related to P, Q, and M in the form satisfying $N = (I - QP)M^{-T}$. We define $Y := C_cN^T$ and $Z := MB_c$. Then, after some algebra, it can be shown that (11) is equivalent to

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{12}^T & F_{22} & F_{23} \\ F_{13}^T & F_{23}^T & \gamma^2 I \end{bmatrix} < 0, \qquad \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{12}^T & H_{22} & H_{23} & 0 \\ H_{13}^T & H_{23}^T & -\gamma^2 I & 0 \\ H_{14}^T & 0 & 0 & -I \end{bmatrix} < 0, \quad (13)$$

where $F_{11} = PA + ZC_y + (PA + ZC_y)^T + C_z^TC_z$, $F_{12} = PB_p + ZD_{yp} + A^TC_q^T\Lambda + C_q^TT$, $F_{13} = PB_w + ZD_{yw}$, $F_{22} = \Lambda C_qB_p + (\Lambda C_qB_p)^T - 2T$, $F_{23} = \Lambda C_qB_w$, $H_{11} = AQ + B_uY + (AQ + B_uY)^T$, $H_{12} = B_p + (AQ + B_uY)^TC_q^T\Lambda + QC_q^TT$, $H_{13} = B_w$, $H_{14} = (C_zQ + D_{zu}Y)^T$, $H_{22} = \Lambda C_qB_p + (\Lambda C_qB_p)^T - 2T$, and $H_{23} = \Lambda C_qB_w$.

By the Completion Lemma [18], for every $Q > 0$, $P \geq Q^{-1}$, the lower right $n \times n$ block of $\tilde{P}$ and that of $\tilde{Q}$ in (12) can be shown to satisfy $R = M^T(P - Q^{-1})^{-1}M$ and $S = N^T(Q - P^{-1})^{-1}N$ respectively. The conditions $\tilde{P} > 0$ and $\tilde{P}\tilde{Q} = I$ with $\tilde{P}$ written in (12) imply

$$\begin{bmatrix} P & I \\ I & Q \end{bmatrix} \geq 0. \quad (14)$$

Restricting (14) to be positive definite, we are effectively searching for full-order controllers (*i.e.*, of order n) [15]. We observe that the second matrix inequality in (13) is BMI, *i.e.*, there are product terms involving (Q, Y) and (Λ, T). This is a direct consequence of optimizing both the compensator parameters (related to Q and Y) and the analysis multiplier (Λ, T) simultaneously. Note that if Λ and T are fixed, then (13) is an LMI in Q and Y. Similarly, if Q and Y are fixed, then (13) is an LMI in Λ and T. The positive definite constraint of $\tilde{P}$ and $\tilde{Q}$ is implied from the existence of symmetric matrices W and X such that

$$\begin{bmatrix} X & Z^T & 0 & 0 \\ Z & P & I & 0 \\ 0 & I & Q & Y^T \\ 0 & 0 & Y & W \end{bmatrix} > 0. \quad (15)$$

In summary, after eliminating A_c from the formulation the optimization problem (9) is equivalent to:

$$\begin{array}{ll} \text{minimize} & \gamma^2 \\ \text{subject to} & (13),\ (15),\ \Lambda \geq 0,\ T \geq 0. \end{array} \quad (16)$$

3.2 Controller Reconstruction

Given that there exist γ^2, P, Q, Y, Z, W, X, Λ and T satisfying (16), we can construct a controller as follows. First we construct the quadratic term of the Lyapunov function, *i.e.*, $\tilde{P}$, such that the condition (8) holds. The set of the quadratic part of the closed-loop Lyapunov functions is parameterized by Eq. (12), where M is an arbitrary invertible matrix. Because M corresponds to a change of coordinates in the controller states x_c, the choice of M has no effect on the controller transfer function [15]. After constructing the Lyapunov function, the set of input/output controller matrices (B_c and C_c) can be parameterized by $B_c = M^{-1}Z$ and $C_c = Y(I - PQ)^{-1}M$. With γ^2, $\tilde{P}$, Λ, T, B_c, and C_c determined, it suffices to find A_c satisfying the condition (10), which can then be formulated as an LMI problem in A_c.

3.3 Algorithm

It has already been shown that BMI problems are NP-hard, and it is thought to be rather unlikely that there is a polynomial time algorithm to compute the optimal solutions [19]. Since there are product terms involving compensator parameters and the Popov parameters, our approach to solving the non-convex optimization problem is based on an iterative procedure. The proposed algorithm, which we call the V–K iteration, basically alternates between three different LMI problems, *i.e.*, (9) with fixed compensator parameters, (16) with fixed multiplier parameters, and (10). The first LMI problem, considered as the *V step* or *analysis step*, is to solve (9) with fixed compensator parameters (A_c, B_c, and C_c) which yields Popov multiplier parameters (Λ and T). For the *K step* or *synthesis step*, the second and third LMI problems are solved. The solution parameters of the second LMI problem, *i.e.*, (16) with fixed multiplier parameters, implicitly includes the input/output compensator matrices (B_c and C_c) as variables. After obtaining B_c and C_c, the dynamics of the compensator A_c can be computed by solving the third LMI problem (10). At this point a robust compensator, which guarantees the robust stability and satisfies the upper bound of the $\mathcal{L}_2$ gain, is completely calculated. We then repeat the procedure until the decrease in the upper bound of the $\mathcal{L}_2$ gain is sufficiently small. The solution algorithm to design a set of controllers with increasing robustness is briefly summarized as the following:

1. Initialize the sector bound nonlinearities to be zero (a nominal system) and design the controller via $\mathcal{H}_\infty$ controller synthesis.
2. Initialize Λ and T by by solving (9) where (A_c, B_c, C_c) are fixed.
3. Repeat{ [Outer Loop]
   (a) Repeat{ [Inner Loop]
      i. Solve the optimization problem (16) for (γ^2, P, Q, Y, Z, W, X) where (Λ,T) are fixed. Then compute $\tilde{P}, B_c$, and C_c by the Completion Lemma.
      ii. Compute A_c by solving a feasibility LMI problem (10).
      iii. Compute Λ and T by by solving (9) where (A_c, B_c, C_c) are fixed.

      } [Inner Loop] Until stopping criterion satisfied.
   (b) Increase the sector bound nonlinearity to the next desired size and initialize Λ and T by the most recent values.

   } [Outer Loop] Until the desired robustness is achieved or the problem is infeasible.

Remark 1. We note that this algorithm has been successfully applied to the parametric robust $\mathcal{H}_2$ control design problem with Popov multipliers [10] and generalized multipliers [11]. Although the robust performance metrics are different, the same solution procedure based on LMI synthesis is very effective and efficient for parametric robust $\mathcal{H}_\infty$ control design. To be specific, the core algorithm is developed in two key steps. First, we eliminate the controller matrix A_c from the problem formulation (9). Then, we solve for the remaining variables which are subsequently used to reconstruct the controller parameters. The V–K iteration is then used to compute the controller parameters. This shows a unique versatility of our solution algorithm for designing robust controllers, and may eventually lead to better insight into the relationship between these synthesis approaches.

Remark 2. The procedure of alternating between the LMI problems is an iterative approach of solving a non-convex optimization problem, which is not guaranteed to converge in general. However, the same algorithm solving the parametric robust $\mathcal{H}_2$ synthesis has been analyzed for several examples in Ref. [20]. These results show that, to the best of our knowledge, the algorithm does converge to the global optimal solution for the simple examples considered. However, much further analysis is required to generalize this statement. Each step of the iteration can be solved very efficiently by a previously developed semidefinite programming algorithm SP [16] and very easily coded using a user-friendly interface SDPSOL [17].

Remark 3. An important distinction between the V–K iteration and the D–K iteration of the μ/K_m synthesis is that in our approach there are shared variables between each iteration: specifically, γ^2 and $\tilde{P}$ are the common variables between the V step and K step (where $\tilde{P}$ appears as P, Q, Y, Z, W and X). However, for the D–K iteration, the D step (the robust analysis with or without curve fitting) is entirely separate from the K step (the $\mathcal{H}_\infty$ synthesis). We conjecture that these shared variables play a key role in the efficiency and robustness of the convergence of this new algorithm to a local optimum, and are currently investigating this point further.

4 Numerical Example

The parametric $\mathcal{H}_\infty$ control design algorithm is performed on the flexible structural benchmark problem, which was previously considered for robust $\mathcal{H}_2$ control design [21, 10, 11]. The system is a cantilevered Bernoulli Euler beam with unit length and mass density, and stiffness scaled so that the fundamental frequency is 1 rad/sec. The infinite order dynamics of the beam are truncated at four modes, where $w_1 = 1$ rad/sec, $w_2 = 6.27$ rad/sec, $w_3 = 17.55$ rad/sec, $w_4 = 34.39$ rad/sec and damping $\zeta = 0.01$. The changes in the system dynamics due to perturbations in the frequency of the third mode cause substantial variations in the system gain and phase in the $17 - 25$ rad/sec frequency range [21, 10]. The disturbance input, control input, sensor output and performance output are all collocated at the tip of the beam, and the frequency of the third mode of the system is considered to be uncertain. Note that this problem was chosen so that the realness of the parametric uncertainty in the design problem is accentuated. To proceed, a frequency dependent weighting function, $W(s)$, was placed on the disturbance to the performance loop. For this study, we choose $W(s) = 10(s + 40)/(s + 400)$.

Several controllers were designed by the Popov controller synthesis (PCS) approach described in §2.2. The controllers were

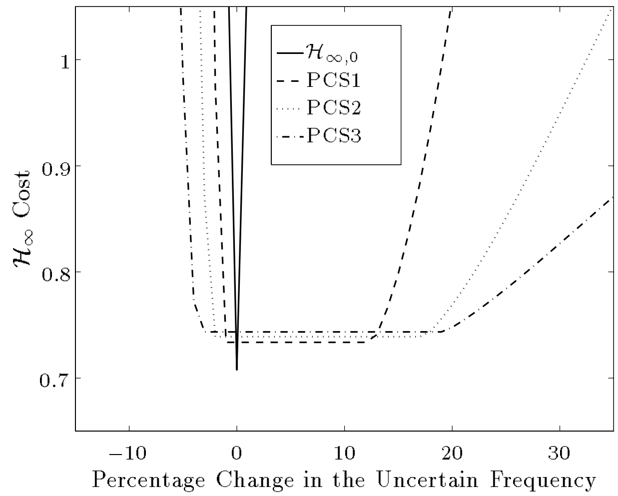

Figure 2: Robust performance plots for Popov controllers with the guaranteed stability bounds ±1, ±2, and ±3% and the nominal $\mathcal{H}_\infty$ controller.

designed for different levels of the frequency uncertainty (*i.e.*, ±1%, ±2%, and ±3%). With this reliable design technique, it is now feasible to undertake a comparison of the Popov controllers with the standard $\mathcal{H}_\infty$ control technique. The curves in Figure 2 are developed by computing the $\mathcal{H}_\infty$ cost for the system with the given percentage change in the mode frequency. The controllers were designed using symmetric sector bounds, with the sizes (*i.e.*, ±1%) given in the figure legend. We first note that the standard $\mathcal{H}_\infty$ controller, labeled by $\mathcal{H}_{\infty,0}$ in the figure, performs extremely well at the nominal frequency. However, it is clearly not robust to changes in the uncertain frequency because the $\mathcal{H}_\infty$ controller was designed without a robust guarantee on the frequency change. As expected, for the Popov controllers the system is robustified to the parameter changes with a slight trade-off on the nominal performance. The plot illustrates that the guaranteed stability boundaries for Popov controllers are obtained and that the actual performance is quite flat and asymmetric about the nominal frequency. This asymmetry was also observed in robust $\mathcal{H}_2$ control designs (see Refs. [10, 11]). Table 1 summarizes the key points for the robust performance analysis of this plot: the percentage change of the nominal $\mathcal{H}_\infty$ cost (*i.e.*, the $\mathcal{H}_\infty$ cost evaluated on the nominal system) for the Popov controllers compared with the nominal $\mathcal{H}_\infty$ cost for the $\mathcal{H}_\infty$ controller, and the lower (upper) achieved and guaranteed stability bounds. The achieved robust stability was determined by analyzing the closed-loop eigenvalues. Similar results are presented in Ref. [10] for the robust $\mathcal{H}_2$ case

Table 1: Robust stability and $\mathcal{H}_\infty$ performance for Popov controllers with various robustness bounds.

Type of Control Design	% Change of Nominal $\mathcal{H}_\infty$ Cost	Lower Bound of Stability %		Upper Bound of Stability %	
		Ach.	Guar.	Guar.	Ach.
$\mathcal{H}_{\infty,0}$	0	−2	0	0	2
PCS1	3.43	−4	−1	1	63
PCS2	4.20	−7	−2	2	80
PCS3	4.86	−12	−3	3	100

Table 2: Achieved robust stability and $\mathcal{H}_\infty$ performance for the robust $\mathcal{H}_\infty$ control, μ/K_m synthesis, and Popov controller synthesis with ±2% guaranteed robust bounds.

Type of Control Design	% Change of Nominal $\mathcal{H}_\infty$ Cost	Lower Stability Bound %	Upper Stability Bound %
$\mathcal{H}_\infty$	67.86	−45	> 400
μ/K_m	21.45	−45	54
PCS	4.20	−7	80

using weighting values that yield very similar loop properties (see detailed discussion in Ref. [21]). A thorough comparison of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers is quite difficult, but a comparison of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance plots illustrates some interesting relationships. The first observation is that the robust performance results are very similar with very flat bottomed asymmetric curves. Furthermore, the changes in nominal performance are quite similar (*i.e.*, relatively small), and both approaches show quite large gaps between the guaranteed and achieved stability bounds. The last characteristic is considered as a function of the conservatism in the Popov robust $\mathcal{H}_\infty$ performance analysis, which has been improved for the $\mathcal{H}_2$ case in Ref. [11]. Thus, using the core algorithm in §2.2 we can design parametric robust $\mathcal{H}_2$ or $\mathcal{H}_\infty$ controllers that yield a consistent closed-loop robust performance.

We continue this discussion by directly comparing controllers from three different techniques: the robust $\mathcal{H}_\infty$ control design, the μ/K_m-synthesis via the D–K iteration with first order scaling transfer function, and the Popov controller synthesis (PCS) using LMI synthesis. These controllers are designed to guarantee the robust stability within ±2% changes of the third mode frequency. The $\mathcal{H}_\infty$ cost of the system with various percentage changes in the mode frequency for these controllers is shown in Figure 3. We first note that the robust $\mathcal{H}_\infty$ compensator assumes a full complex block of combined uncertainty and performance in the design methodology and its controller order is equal to the order of the nominal LTI system plus that of the weighting function ($8 + 1 = 9$). On the other hand, the μ/K_m synthesis exploits the structure of the uncertainty and performance blocks. As a consequence, it produces a higher order of the controller, *i.e.*, the order of the nominal LTI system augmented with the weighting function plus the order of first order scaling function and its inverse ($9 + (2 \times 3) = 15$). The figure shows that all robust controllers achieve the desired stability bounds and that in the desired uncertainty region the Popov synthesis yields an improved $\mathcal{H}_\infty$ performance, which is much lower than that of other approaches. Moreover, the actual performance for all controllers is quite flat in this region of guaranteed robustness. Table 2 summarizes the key points for the achieved robust stability bounds and performance for this figure. For the Popov synthesis, the achieved lower stability bound is much closer to the lower guaranteed robustness bound and much smaller than that of the other designs. However, the achieved upper robustness bound is slightly larger than that of the μ/K_m synthesis and much smaller comparing to the upper bound of the robust $\mathcal{H}_\infty$ design. This robust stability and $\mathcal{H}_\infty$ performance analysis provide insights into the issue of conservatism for these design techniques. Although the

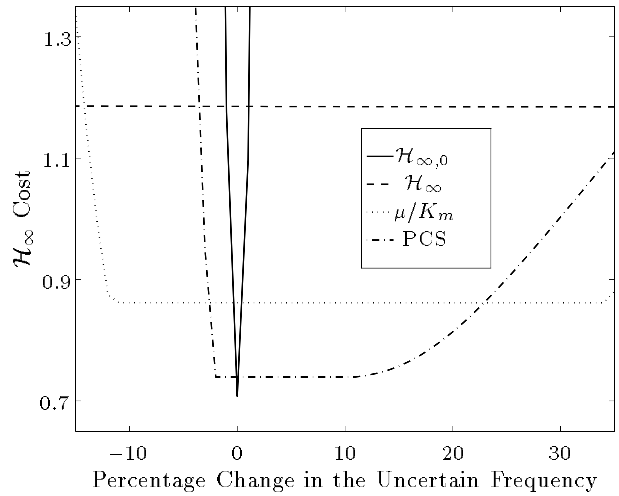

Figure 3: Robust performance plots for the robust $\mathcal{H}_\infty$ control, μ/K_m synthesis, and Popov controller synthesis with the guaranteed stability bounds equal $\pm 2\%$. The performance analysis for the nominal $\mathcal{H}_\infty$ controller is given as a reference.

μ/K_m controller is appropriate for systems with structured complex uncertainty, it obviously becomes conservative for the case of real parametric uncertainty. Thus, the Popov controller synthesis provides a means to capture the realness of the uncertainty which results in a reduction of the conservatism in the control design.

5 Conclusions

This paper presents an efficient and effective design technique for the real parametric μ/K_m synthesis problem by applying the Popov robust $\mathcal{H}_\infty$ performance analysis to a Lur'e system. As discussed, this approach offers several potential benefits over the current D–K iteration and the BMI synthesis procedure. A unique feature of our approach is that the core solution algorithm can be used to solve both the parametric robust $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems. We first illustrate this approach by using the algorithm to design robust controllers for a simple system with real parametric uncertainty. A direct comparison of this approach with other robust $\mathcal{H}_\infty$ control design techniques for systems with mixed uncertainty, such as μ/K_m synthesis, indicates that the Popov controller synthesis yields less conservative designs.

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