Region of Attraction Comparison for Gradient Projection Anti-windup Compensated Systems

Justin Teo\textsuperscript{1}  Jonathan P. How\textsuperscript{2}

\textsuperscript{1}DSO National Laboratories, Singapore

\textsuperscript{2}Aerospace Controls Laboratory
Department of Aeronautics & Astronautics
Massachusetts Institute of Technology

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Motivation

“Conventional” Anti-windup Problem

Given nominal plant and controller
“Conventional” Anti-windup Problem

Given nominal plant and controller, design anti-windup compensator such that [Tarbouriech and Turner 2009]:

- unconstrained response recovered when no controls saturate
- performance improved when some controls saturate
- region of attraction (ROA) *maximized* when $r \equiv 0$
“Conventional” Anti-windup Problem

Given nominal plant and controller, design anti-windup compensator such that [Tarbouriech and Turner 2009]:
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- performance improved when some controls saturate
- region of attraction (ROA) \textbf{maximized} when \( r \equiv 0 \)

\textbf{Question}

Can some solutions (corresponding to some initial conditions) that were stable (uncompensated) become unstable with anti-windup?
Motivation

Maximizing ROA Not Adequate

\[ \text{ROA: } R_{au} \]

\[ \text{ROA: } R_n \]

Want \( R_{aw} \supset R_n \) or \( R_{aw} \supset \Omega \), where \( \Omega \subset R_n \) is an ROA estimate

Results for GPAW compensated planar LTI systems available [Teo and How 2010a,c]. Develop similar results for MIMO nonlinear systems.
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Results for GPAW compensated planar LTI systems available [Teo and How 2010a,c]. Develop similar results for MIMO nonlinear systems
Overview of GPAW Compensation

- References [Teo and How 2009, 2010a,b,c,d, Teo 2011]
- Gradient Projection Anti-windup (GPAW) scheme constructed for saturated nonlinear MIMO plants driven by MIMO nonlinear controllers

\[\dot{x}_c = f_c(x_c, y, r)\]
\[u_c = g_c(x_c)\]
\[\text{Projection operator } R_I^* \text{ projects controller state onto unsaturated region} \]
\[K := \{x | \text{sat}(g_c(x)) = g_c(x)\}\]

Achieves "controller state-output consistency", a unique property:
\[\text{sat}(u_g) \equiv u_g \text{ or } \text{sat}(g_c(x_g)) \equiv g_c(x_g)\]

Extension of "conditional integration" method [Fertik and Ross 1967]
Can be realized in 3 equivalent ways (closed-form expressions also)
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- Gradient Projection Anti-windup (GPAW) scheme constructed for saturated nonlinear MIMO plants driven by MIMO nonlinear controllers

- For “strictly proper” nonlinear controllers
  \[ \dot{x}_c = f_c(x_c, y, r) \quad \text{GPAW, } \Gamma = \Gamma^T > 0 \quad \dot{x}_g = R_{\mathcal{I}*}(x_g, y, r)f_c(x_g, y, r) \]
  \[ u_c = g_c(x_c) \quad u_g = g_c(x_g) \]

- Projection operator \( R_{\mathcal{I}*} \) projects controller state onto unsaturated region \( K := \{ x \mid \text{sat}(g_c(x)) = g_c(x) \} \)
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- Extension of “conditional integration” method [Fertik and Ross 1967]
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GPAW Overview

GPAW Scheme Visualization

Nominal controller:
\[ \dot{x}_c = f_c(x_c, y, r) \]
\[ u_c = g_c(x_c) \]

GPAW controller:
\[ \dot{x}_g = R_I \ast f_c(x_g, y, r) \]
\[ u_g = g_c(x_g) \]

Boundaries:
\[ H_1, H_2, G_3 \]

Gradients:
\[ \nabla h_i(x_g) = \pm \nabla g_{ci}(x_g) \]

Unsaturated region:
\[ K := \{ \bar{x} \mid \text{sat}(g_c(\bar{x})) = g_c(\bar{x}) \} \]

Nominal update:
\[ f_{ci} := f_c(x_{gi}, y(t_i), r(t_i)) \text{ for } x_{gi} := x_g(t_i) \]

Projections:
\[ f_{gi} := R_I \ast f_{ci} \]
General ROA Comparison Result

- **System:** \( \dot{x} = f(x) \), equilibrium: \( x_{eq} \), solution from \( x_0 \): \( \phi(t, x_0) \)
- **ROA:** \( R_A(x_{eq}) := \{ x \mid \lim_{t \to \infty} \phi(t, x) = x_{eq} \} \)
- **ROA estimate:** \( \Omega \subset R_A(x_{eq}) \)
- **ROA estimate** \( \Omega_V \) associated with Lyapunov function, \( V \):
  \[
  V(x_{eq}) = 0, \quad V(x) > 0, \quad \forall x \in \Omega_V \setminus \{ x_{eq} \}
  \]
  \[
  \dot{V}(x) = \frac{\partial V(x)}{\partial x} f(x) \leq -\alpha(\| x - x_{eq} \|), \quad \forall x \in \Omega_V
  \]
- \( M \) is positively invariant if forward solution starting in \( M \) stays in \( M \)
General ROA Comparison Result

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- $M$ is positively invariant if forward solution starting in $M$ stays in $M$

**Lemma (ROA Comparison for General Systems)**

Assume $x_{eq}$ is asymptotically stable equilibrium for two autonomous systems $\dot{x} = f_1(x)$ and $\dot{x} = f_2(x)$. Let $R_{A1}(x_{eq})$ and $R_{A2}(x_{eq})$ be associated ROAs. Let $\Omega_V \subset R_{A1}(x_{eq})$ be ROA estimate associated with Lyapunov function $V$. If $\Omega_2 \subset \Omega_V$ is positively invariant for $\dot{x} = f_2(x)$ and

$$\frac{\partial V(x)}{\partial x} f_2(x) \leq \frac{\partial V(x)}{\partial x} f_1(x)$$

for all $x \in \Omega_2$, then $\Omega_2 \subset R_{A2}(x_{eq})$
An ROA Comparison Result

<table>
<thead>
<tr>
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<th>GPAW Controller, $\Sigma_{gpaw}$</th>
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<td>$\dot{x} = f(x, \text{sat}(u))$</td>
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<td>$y = g(x, \text{sat}(u))$</td>
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Assume $z_{eq} \in \mathbb{R}^n \times (K \setminus \partial K)$ is asymptotically stable for $\Sigma_n$ and $\Sigma_g$

Nominal System: $\Sigma_n : \Sigma_p + \Sigma_c$  
GPAW System: $\Sigma_g : \Sigma_p + \Sigma_{gpaw}$  

ROA: $R_n(z_{eq})$  
ROA: $R_g(z_{eq})$
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\dot{x} = f(x, \text{sat}(u))
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\dot{x}_c = f_c(x_c, y)
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u = g_c(x_c)
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GPAW Controller, $\Sigma_{gpaw}$
\[
\dot{x}_g = R_I f_c(x_g, y)
\]
\[
u = g_c(x_g)
\]

ROA estimate:
\[
\Omega_V = \{ \bar{z} \mid V(\bar{z}) \leq c \} \subset R_n(\bar{z}_{eq})
\]

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Nominal System: $\Sigma_n : \Sigma_p + \Sigma_c$
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\dot{x}_c = f_c(x_c, y) \\
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GPAW Controller, $\Sigma_{gpaw}$
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\dot{x}_g = R_I^* f_c(x_g, y) \\
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Assume $z_{eq} \in \mathbb{R}^n \times (K \setminus \partial K)$ is asymptotically stable for $\Sigma_n$ and $\Sigma_g$

Nominal System: $\Sigma_n: \Sigma_p + \Sigma_c$

GPAW System: $\Sigma_g: \Sigma_p + \Sigma_{gpaw}$

ROA estimate: $\Omega_V = \{ \bar{z} | V(\bar{z}) \leq c \} \subset R_n(z_{eq})$ for some Lyapunov function $V(z) = V(x, x_c)$ of $\Sigma_n$

Theorem (ROA Bounds for GPAW Compensated System)

If there exists a $\Gamma = \Gamma^T > 0$ such that
\[
\frac{\partial V(\bar{x}, \bar{x}_c)}{\partial x_c} R_I^* f_c \leq \frac{\partial V(\bar{x}, \bar{x}_c)}{\partial x_c} f_c, \quad \forall (\bar{x}, \bar{x}_c) \in \Omega_V \cap (\mathbb{R}^n \times K)
\]

then $\Sigma_g$ with $\Gamma$ has ROA satisfying $(\Omega_V \cap (\mathbb{R}^n \times K)) \subset R_g(z_{eq})$
Observations on ROA Comparison

- ROA comparison is relative to uncompensated system
- If $\Omega_V = R_n(z_{eq})$, conclusion is $R_n(z_{eq}) \cap (\mathbb{R}^n \times K) \subset R_g(z_{eq})$
Observations on ROA Comparison

- ROA comparison is **relative** to uncompensated system
- If $\Omega_V = R_n(z_{eq})$, conclusion is $R_n(z_{eq}) \cap (\mathbb{R}^n \times K) \subset R_g(z_{eq})$
- States loosely that ROA of $\Sigma_g$ is “not less than” ROA estimate $\Omega_V$ (or ROA $R_n(z_{eq})$)
- Specialized with additional assumptions (e.g. LTI) – yields LMI conditions [Teo 2011]
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- Specialized with additional assumptions (e.g. LTI) – yields LMI conditions [Teo 2011]
- Main condition: $
\frac{\partial V(\bar{x},\bar{x}_c)}{\partial x_c} R_I^* f_c \leq \frac{\partial V(\bar{x},\bar{x}_c)}{\partial x_c} f_c$
  independent of $\text{sat}(\cdot)$
- Can be used in two ways: comparison against ROA estimate of unconstrained system or nominal system
- Result is applicable to MIMO nonlinear systems, but likely conservative
Application of ROA Comparison Result

Example nonlinear planar system [Khalil 2002]

\[ \Sigma_n: \begin{align*}
\dot{x} &= -\text{sat}(u) \\
\dot{u} &= x + (x^2 - 1)u
\end{align*} \]

\[ \Sigma_{gs}: \begin{align*}
\dot{x} &= -\text{sat}(u) \\
\dot{u} &= \begin{cases} 
0, & \text{if } A_1 \\
x + (x^2 - 1)u, & \text{otherwise}
\end{cases}
\end{align*} \]
Example nonlinear planar system [Khalil 2002]

\[
\Sigma_u: \begin{cases} 
\dot{x} = -u \\
\dot{u} = x + (x^2 - 1)u 
\end{cases} \quad \Sigma_g: \begin{cases} 
\dot{x} = -u \\
\dot{u} = \begin{cases} 0, & \text{if } A_1 \\
x + (x^2 - 1)u, & \text{otherwise} \end{cases} 
\end{cases}
\]

Compare with ROA estimate \( \Omega_V \) of unconstrained system:

\[\phi_u(t, z_0), \phi_{gu}(t, z_0)\]
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Compare with ROA estimate \( \Omega_V \) of \textit{unconstrained} system:

Toy example \textbf{defeats} methods for LTI systems, feedback linearizable systems, and \textit{nonlinear anti-windup} [Morabito et al. 2004]
References I


Conclusions and Future Work

- Maximizing ROA not adequate - want $R_n \subset R_{aw}$
- Developed ROA comparison result for general systems
- Specialized to GPAW compensated saturated MIMO nonlinear systems
- Toy example defeats 3 classes of state-of-the-art anti-windup methods
- Current result still likely conservative
- Plenty of work to find qualitatively similar but less conservative results
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For more information on GPAW compensation

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Questions?
ROAs of Saturated Planar LTI Systems, $R_n \subset R_g$

Unstable plant, stable controller

Symmetric constraints

Asymmetric constraints

Stable plant, unstable controller
ROA Comparison Indicates True Advantage

- Existing anti-windup results are in “absolute” sense
  - may not indicate any advantages of anti-windup scheme

- ROA comparison result is in “relative” sense
  - directly shows advantage of GPAW scheme
  - first in new anti-windup paradigm

\[
\begin{align*}
\text{States} \ &\text{loosely that } ROA \ of \\
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\Omega \ &\text{V}
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Applies for asymmetric saturation constraints

Specialized with additional assumptions (e.g. LTI)

Main condition:

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\frac{\partial V(\bar{x}, \bar{x}_c)}{\partial x_c} R_I^* f_c \leq \frac{\partial V(\bar{x}, \bar{x}_c)}{\partial x_c} f_c
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- Applies for **asymmetric** saturation constraints
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- Main condition: $\frac{\partial V(\bar{x},\bar{x}_c)}{\partial x_c} R^* f_c \leq \frac{\partial V(\bar{x},\bar{x}_c)}{\partial x_c} f_c$ independent of sat($\cdot$)

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