Region of Attraction Comparison for Gradient Projection Anti-windup Compensated Systems

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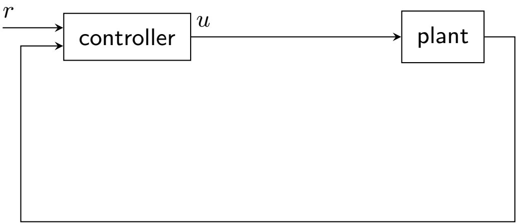
Outline

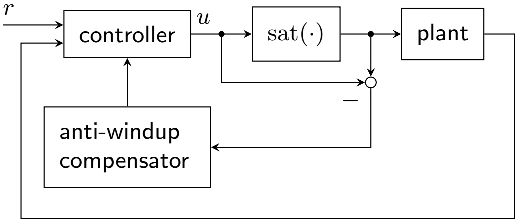
1. Motivation
2. GPAW Overview
3. General ROA Comparison
4. ROA Comparison for GPAW Compensated Systems
5. Conclusions and Future Work

"Conventional" Anti-windup Problem



Given nominal plant and controller

"Conventional" Anti-windup Problem



Given nominal plant and controller, design anti-windup compensator such that [Tarbouriech and Turner 2009]:

- unconstrained response recovered when no controls saturate
- performance improved when some controls saturate
- region of attraction (ROA) **maximized** when $r \equiv 0$

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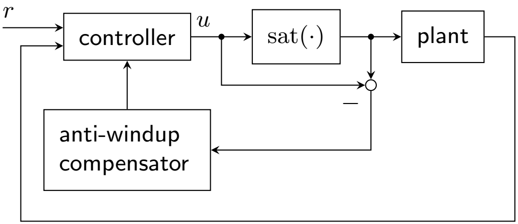

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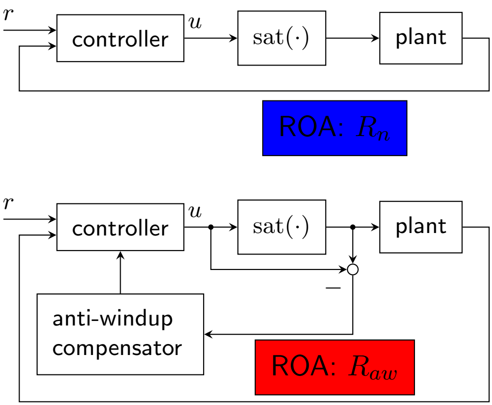
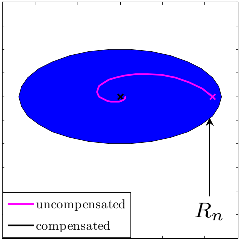
Question

Can some solutions (corresponding to some initial conditions) that were stable (uncompensated) become unstable with anti-windup?

Maximizing ROA Not Adequate





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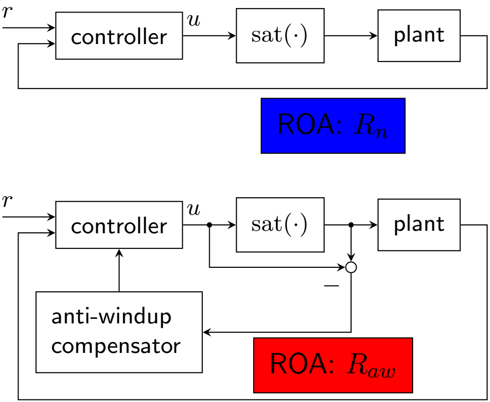

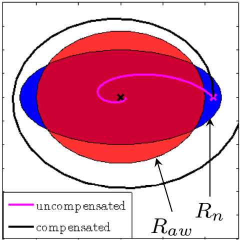

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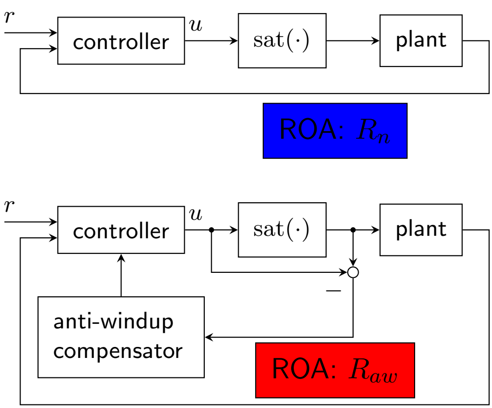

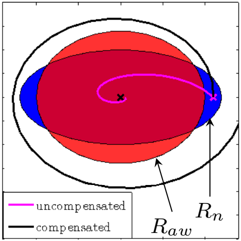

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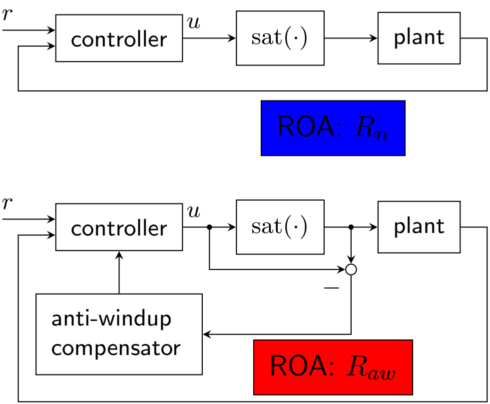

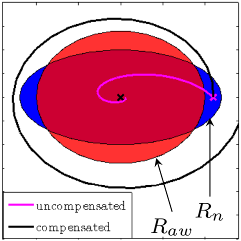

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Results for GPAW compensated planar LTI systems available [Teo and How 2010a,c]. Develop similar results for MIMO nonlinear systems

Overview of GPAW Compensation

- References [Teo and How 2009, 2010a,b,c,d, Teo 2011]
- Gradient Projection Anti-windup (GPAW) scheme constructed for saturated nonlinear MIMO plants driven by MIMO nonlinear controllers

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- For "strictly proper" nonlinear controllers

$$\begin{aligned} \dot{x}_c &= f_c(x_c, y, r) \\ u_c &= g_c(x_c) \end{aligned} \quad \xrightarrow{\text{GPAW},\Gamma=\Gamma^{\mathrm{T}}>0} \quad \begin{aligned} \dot{x}_g &= R_{\mathcal{I}^*}(x_g, y, r) f_c(x_g, y, r) \\ u_g &= g_c(x_g) \end{aligned}$$

- Projection operator $R_{\mathcal{I}^*}$ projects controller state onto unsaturated region $K := \{x \mid \mathrm{sat}(g_c(x)) = g_c(x)\}$

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- Projection operator $R_{\mathcal{I}^*}$ projects controller state onto unsaturated region $K := \{x \mid \mathrm{sat}(g_c(x)) = g_c(x)\}$
- Achieves "controller state-output consistency", a **unique property**:

$$\mathrm{sat}(u_g) \equiv u_g \quad \text{or} \quad \mathrm{sat}(g_c(x_g)) \equiv g_c(x_g)$$

- Extension of "conditional integration" method [Fertik and Ross 1967]
- Can be realized in 3 equivalent ways (closed-form expressions also)

GPAW Scheme Visualization

Nominal controller:
$\dot{x}_c = f_c(x_c, y, r)$
$u_c = g_c(x_c)$

GPAW controller:
$\dot{x}_g = R_{\mathcal{I}^*} f_c(x_g, y, r)$
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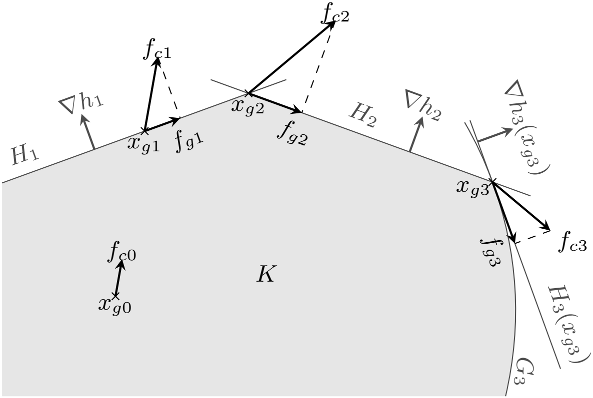
Boundaries:
H_1, H_2, G_3

Gradients:
$\nabla h_i(x_g) = \pm \nabla g_{ci}(x_g)$



Unsaturated region: $K := \{\bar{x} \mid \mathrm{sat}(g_c(\bar{x})) = g_c(\bar{x})\}$
Nominal update: $f_{ci} := f_c(x_{gi}, y(t_i), r(t_i))$ for $x_{gi} := x_g(t_i)$
Projections: $f_{gi} := R_{\mathcal{I}^*} f_{ci}$

General ROA Comparison Result

- System: $\dot{x} = f(x)$, equilibrium: x_{eq}, solution from x_0: $\phi(t, x_0)$
- ROA: $R_A(x_{eq}) := \{x \mid \lim_{t\to\infty} \phi(t,x) = x_{eq}\}$
- ROA **estimate**: $\Omega \subset R_A(x_{eq})$
- ROA estimate Ω_V associated with Lyapunov function, V:

$$V(x_{eq}) = 0, \qquad V(x) > 0, \quad \forall x \in \Omega_V \setminus \{x_{eq}\}$$
$$\dot{V}(x) = \frac{\partial V(x)}{\partial x} f(x) \leq -\alpha(\|x - x_{eq}\|), \quad \forall x \in \Omega_V$$

- M is positively invariant if forward solution starting in M stays in M

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Lemma (ROA Comparison for General Systems)

Assume x_{eq} is asymptotically stable equilibrium for two autonomous systems $\dot{x} = f_1(x)$ and $\dot{x} = f_2(x)$. Let $R_{A1}(x_{eq})$ and $R_{A2}(x_{eq})$ be associated ROAs. Let $\Omega_V \subset R_{A1}(x_{eq})$ be ROA estimate associated with Lyapunov function V. If $\Omega_2 \subset \Omega_V$ is positively invariant for $\dot{x} = f_2(x)$ and $\frac{\partial V(x)}{\partial x} f_2(x) \leq \frac{\partial V(x)}{\partial x} f_1(x)$ for all $x \in \Omega_2$, **then** *$\Omega_2 \subset R_{A2}(x_{eq})$*

An ROA Comparison Result

Plant, Σ_p	Nominal Controller, Σ_c	GPAW Controller, Σ_{gpaw}
$\dot{x} = f(x, \text{sat}(u))$	$\dot{x}_c = f_c(x_c, y)$	$\dot{x}_g = R_{\mathcal{I}^*} f_c(x_g, y)$
$y = g(x, \text{sat}(u))$	$u = g_c(x_c)$	$u = g_c(x_g)$

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Assume $z_{eq} \in \mathbb{R}^n \times (K \setminus \partial K)$ is asymptotically stable for Σ_n and Σ_g

Nominal System: $\Sigma_n : \Sigma_p + \Sigma_c$ ROA: $R_n(z_{eq})$

GPAW System: $\Sigma_g : \Sigma_p + \Sigma_{gpaw}$ ROA: $R_g(z_{eq})$

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ROA estimate: $\Omega_V = \{\bar{z} \mid V(\bar{z}) \leq c\} \subset R_n(z_{eq})$ for some Lyapunov function $V(z) = V(x, x_c)$ of Σ_n

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Theorem (ROA Bounds for GPAW Compensated System)

If there exists a $\Gamma = \Gamma^{\mathrm{T}} > 0$ *such that*

$$\frac{\partial V(\bar{x}, \bar{x}_c)}{\partial x_c} R_{\mathcal{I}^*} f_c \leq \frac{\partial V(\bar{x}, \bar{x}_c)}{\partial x_c} f_c, \qquad \forall (\bar{x}, \bar{x}_c) \in \Omega_V \cap (\mathbb{R}^n \times K)$$

then Σ_g *with* Γ *has ROA satisfying* $(\Omega_V \cap (\mathbb{R}^n \times K)) \subset R_g(z_{eq})$

Observations on ROA Comparison

- ROA comparison is **relative** to uncompensated system
- If $\Omega_V = R_n(z_{eq})$, conclusion is $R_n(z_{eq}) \cap (\mathbb{R}^n \times K) \subset R_g(z_{eq})$

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- Can be used in two ways: comparison against ROA estimate of unconstrained system or nominal system
- Result is applicable to MIMO nonlinear systems, but likely conservative

Application of ROA Comparison Result

Example nonlinear planar system [Khalil 2002]

$$\Sigma_n : \begin{cases} \dot{x} = -\operatorname{sat}(u) \\ \dot{u} = x + (x^2 - 1)u \end{cases} \qquad \Sigma_{gs} : \begin{cases} \dot{x} = -\operatorname{sat}(u) \\ \dot{u} = \begin{cases} 0, & \text{if } A_1 \\ x + (x^2 - 1)u, & \text{otherwise} \end{cases} \end{cases}$$

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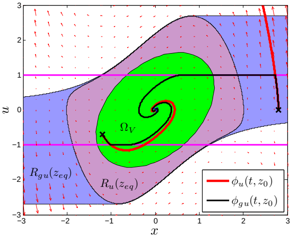
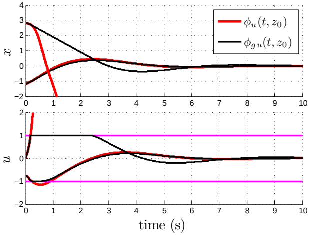
Compare with ROA estimate Ω_V of **unconstrained** system:





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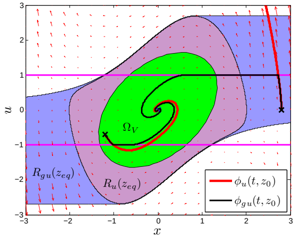

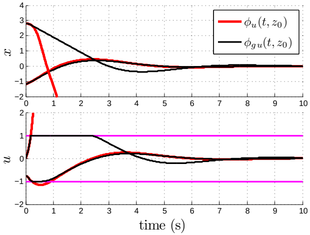

Toy example **defeats** methods for LTI systems, feedback linearizable systems, and nonlinear anti-windup [Morabito et al. 2004]

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Conclusions and Future Work

- Maximizing ROA not adequate - want $R_n \subset R_{aw}$
- Developed ROA comparison result for general systems
- Specialized to GPAW compensated saturated MIMO nonlinear systems
- Toy example defeats 3 classes of state-of-the-art anti-windup methods
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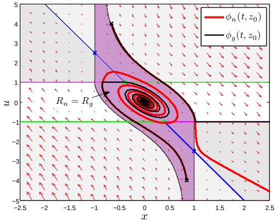
Questions?

ROAs of Saturated Planar LTI Systems, $R_n \subset R_g$

Unstable plant, stable controller

Symmetric constraints



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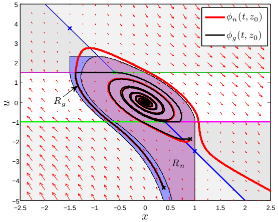

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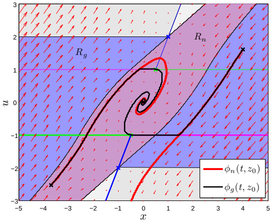

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  - first in new anti-windup paradigm

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