Geometric Properties of Gradient Projection Anti-windup Compensated Systems

Justin Teo and Jonathan P. How

Aerospace Controls Laboratory
Department of Aeronautics & Astronautics
Massachusetts Institute of Technology

American Control Conference
July 2, 2010
1. Introduction

2. Controller State-output Consistency

3. Approximate Nominal Controller

4. Geometric Bounding Condition

5. Conclusions
Motivation

**Well Recognized Fact** [Bernstein and Michel 1995]

Control saturation affects virtually all practical control systems.

Effects called *windup* [Kothare et al. 1994, Edwards and Postlethwaite 1998]. On control saturation,

- Performance degradation (with certainty)
- Instability (possibly)
Motivation

Well Recognized Fact [Bernstein and Michel 1995]
Control saturation affects virtually all practical control systems.


- Performance degradation (with certainty)
- Instability (possibly)

- So far, stability results only for constrained planar LTI systems
- Here, prove state-output consistency and a geometric property
GPAW Scheme Overview

Saturated plant: \( \dot{x} = f(x, \text{sat}(u)), y = g(x, \text{sat}(u)) \). Nominal controller and GPAW compensated controller

\[
\begin{align*}
\dot{x}_c &= f_c(x_c, y, r(t)) \\
u &= g_c(x_c, y, r(t))
\end{align*}
\]

GPAW, \( \Gamma = \Gamma^T > 0 \)

\[
\begin{align*}
\dot{x}_g &= R_{I*} f_c(x_g, y, r(t)) \\
u &= g_c(x_g, y, r(t))
\end{align*}
\]

Everything rests on \( R_{I*} \)!
GPAW Scheme Overview

Saturated plant: \( \dot{x} = f(x, \text{sat}(u)), \quad y = g(x, \text{sat}(u)) \). Nominal controller and GPAW compensated controller

\[
\begin{align*}
\dot{x}_c &= f_c(x_c, y, r(t)) \\
u &= g_c(x_c, y, r(t)) \\
gPAW, \Gamma &= \Gamma^T > 0 \\
\dot{x}_g &= R_{I^*} f_c(x_g, y, r(t)) \\
u &= g_c(x_g, y, r(t))
\end{align*}
\]

Everything rests on \( R_{I^*} \)!

\begin{itemize}
\item \( R_{I^*} \) defined by **online** solution to a combinatorial optimization subproblem or **convex** quadratic program [Teo and How 2010]
\item generalizes **conditional integration method** [Fertik and Ross 1967] using ideas from **gradient projection method** [Rosen 1960, 1961]
\item attempts to maintain **controller state-output consistency**. So far, achieved: \( \text{sat}(u) \approx u \). Proven here: \( \text{sat}(u) = u \), as desired
\end{itemize}
GPAW Scheme Visualization

Nominal controller:

\[
\dot{x}_c = f_c(x_c, y, r(t)) \\
u = g_c(x_c, y, r(t))
\]

GPAW controller:

\[
\dot{x}_g = f_g(x_g, y, r(t)) \\
u = g_c(x_g, y, r(t))
\]

\(H_1, H_2, G_3\) induced by saturation

- Note: \(f_c(x_{gi}) := f_c(x_{gi}, y, r(t))\) and \(f_{gi} := f_g(x_{gi}, y, r(t))\)

- Unsaturated region is

\[
K(y, r(t)) = \{x_g \in \mathbb{R}^q \mid \text{sat}(g_c(x_g, y, r(t))) = g_c(x_g, y, r(t))\}
\]
Controller State-output Consistency

- For output $u = g_c(x_g, y, r(t))$, get $\dot{u} = \frac{\partial g_c}{\partial x_g} \dot{x}_g + \frac{\partial g_c}{\partial y} \dot{y} + \frac{\partial g_c}{\partial r} \dot{r}$

- Previously identified limitation: when $\left\| \frac{\partial g_c}{\partial x_g} \dot{x}_g \right\| \ll \left\| \frac{\partial g_c}{\partial y} \dot{y} + \frac{\partial g_c}{\partial r} \dot{r} \right\|$, then GPAW scheme likely ineffective (only modify controller state)

- Restrict consideration to controllers with output equations **not depending** on $(y, r(t))$, ie.

  $$u = g_c(x_g) \quad \text{and NOT} \quad u \neq g_c(x_g, y, r(t))$$
Controller State-output Consistency

- For output $u = g_c(x_g, y, r(t))$, get $\dot{u} = \frac{\partial g_c}{\partial x_g} \dot{x}_g + \frac{\partial g_c}{\partial y} \dot{y} + \frac{\partial g_c}{\partial r} \dot{r}$

- Previously identified limitation: when $\left| \frac{\partial g_c}{\partial x_g} \dot{x}_g \right| \ll \left| \frac{\partial g_c}{\partial y} \dot{y} + \frac{\partial g_c}{\partial r} \dot{r} \right|$, then GPAW scheme likely ineffective (only modify controller state)

- Restrict consideration to controllers with output equations not depending on $(y, r(t))$, ie.

$$u = g_c(x_g) \quad \text{and NOT} \quad u \neq g_c(x_g, y, r(t))$$

Theorem (Controller Output Consistency)

Consider the GPAW compensated controller whose output equation is of the form $u = g_c(x_g)$. If there exists a $T \in \mathbb{R}$ such that $\text{sat}(u(T)) = u(T)$, then $\text{sat}(u(t)) = u(t)$ holds for all $t \geq T$.

Unique property among anti-windup schemes (less specializations)
Implications

Without controller state-output consistency, closed loop system is

\[
\begin{align*}
\dot{x} &= f(x, \text{sat}(g_c(x_g))) \\
\dot{x}_g &= R_{I*} f_c(x_g, g(x, \text{sat}(g_c(x_g))), r(t))
\end{align*}
\]
Implications

Without controller state-output consistency, closed loop system is

\[ \dot{x} = f(x, \text{sat}(g_c(x_g))) \]
\[ \dot{x}_g = R_{I^*} f_c(x_g, g(x, \text{sat}(g_c(x_g))), r(t)) \]

With controller state-output consistency, \text{sat}(\cdot) \text{ eliminated}

\[ \dot{x} = f(x, g_c(x_g)) \]
\[ \dot{x}_g = R_{I^*} f_c(x_g, g(x, g_c(x_g)), r(t)) \]

provided \( x_g(0) \) initialized such that \( \text{sat}(g_c(x_g(0))) = g_c(x_g(0)) \)
Controller State-output Consistency

Implications

Without controller state-output consistency, closed loop system is

\[ \dot{x} = f(x, \text{sat}(g_c(x_g))) \]
\[ \dot{x}_g = R_{I*} f_c(x_g, g(x, \text{sat}(g_c(x_g))), r(t)) \]

With controller state-output consistency, \text{sat}(\cdot) eliminated

\[ \dot{x} = f(x, g_c(x_g)) \]
\[ \dot{x}_g = R_{I*} f_c(x_g, g(x, g_c(x_g)), r(t)) \]

provided \( x_g(0) \) initialized such that \( \text{sat}(g_c(x_g(0))) = g_c(x_g(0)) \)

- All complications arising from saturation accounted for by \( R_{I*} \)
- Allows stability conditions to be stated in terms of unconstrained dynamics
Approximate Nominal Controller

- Previously restricted to controllers with \( u = g_c(x_c) \). Now consider when \( u = g_c(x_c, y) \) (same construction when \( u = g_c(x_c, y, r(t)) \))

- “Design” unity DC gain, exponentially stable low-pass filter

\[
\dot{y} = a(y - \hat{y}) \quad \hat{y}(0) = y(0)
\]

and replace measurement \( y \) by its approximation only in the output

\[
u = g_c(x_c, y) \quad \implies \quad u = g_c(x_c, \hat{y})
\]

- Approximate nominal controller with augmented state \( \tilde{x}_c := (x_c, \hat{y}) \) is

\[
\begin{align*}
\dot{x}_c &= f_c(x_c, y, r(t)) \\
u &= g_c(x_c, y) \\
a &\gg 1 \quad \approx \quad \Rightarrow \quad \\
\dot{\tilde{x}}_c &= f_c(x_c, y, r(t)) \\
\dot{\hat{y}} &= a(y - \hat{y}) \\
u &= g_c(x_c, \hat{y}) = g_c(\tilde{x}_c)
\end{align*}
\]

- Singular perturbation theory [Khalil 2002] shows that approximation can be made arbitrarily well as \( a \to \infty \)
Let $K$ be unsaturated region,

$$K = \{ x \in \mathbb{R}^q \mid \text{sat}(g_c(x)) = g_c(x) \}$$

Let $f_n(t, z)$, $f_p(t, z)$ be the vector fields of uncompensated ($\dot{z} = f_n(t, z)$) and GPAW compensated systems ($\dot{z} = f_p(t, z)$), $\Gamma$ the GPAW parameter.

Theorem (Geometric Bounding Condition)

If unsaturated region $K \subset \mathbb{R}^q$ is a star domain, then for any $z \in (\mathbb{R}^n \times K)$ and any $z_{ker} \in (\mathbb{R}^n \times \ker(K))$, the geometric condition

$$\langle z - z_{ker}, \tilde{\Gamma}^{-1} f_p(t, z) \rangle \leq \langle z - z_{ker}, \tilde{\Gamma}^{-1} f_n(t, z) \rangle,$$

holds for all $t \in \mathbb{R}$, where $\tilde{\Gamma} = \begin{bmatrix} I & 0 \\ 0 & \Gamma \end{bmatrix}$.
Star Domains

Examples and counterexamples of Star Domains in $\mathbb{R}^2$:

Star Domain, $\ker(X) \neq \emptyset$

- Any convex set $X$ is also a star domain with $\ker(X) = X$
- For any non-convex star domain, $\ker(X)$ is a strict subset of $X$
- If $X$ is a star domain, then $\mathbb{R}^n \times X$ is also a star domain with kernel $\mathbb{R}^n \times \ker(X)$

Not Star Domain, $\ker(X) = \emptyset$
Geometric Interpretation

Closed-loop systems:
Uncompensated,
\[ \dot{z} = f_n(t, z) \]
GPAW compensated,
\[ \dot{z} = f_p(t, z) \]
Implications

Geometric Implications

If a Lyapunov function $V(x, x_c)$ exists for the uncompensated system such that on the boundary of the unsaturated region $\mathbb{R}^n \times K$, $\frac{\partial V}{\partial x_c}$ always points out from the kernel $\ker(K)$, then it is also a Lyapunov function for the GPAW compensated system!
Implications

Geometric Implications

If a Lyapunov function $V(x, x_c)$ exists for the uncompensated system such that on the boundary of the unsaturated region $\mathbb{R}^n \times K$, $\frac{\partial V}{\partial x_c}$ always points out from the kernel $\ker(K)$, then it is also a Lyapunov function for the GPAW compensated system!

Conjecture

That for any Lyapunov function for any uncompensated system, there exists a derived Lyapunov function satisfying above property.
Implications

Geometric Implications

If a Lyapunov function $V(x, x_c)$ exists for the uncompensated system such that on the boundary of the unsaturated region $\mathbb{R}^n \times K$, $\frac{\partial V}{\partial x_c}$ always points out from the kernel $\ker(K)$, then it is also a Lyapunov function for the GPAW compensated system!

Conjecture

That for any Lyapunov function for any uncompensated system, there exists a derived Lyapunov function satisfying above property

Proof of geometric condition in ACC paper is faulty. Corrected in:

Conclusions

- Recalled GPAW scheme characteristics

- Main results:
  - controller state-output consistency
  - geometric bounding condition

- New results:
  - can infer stability for a class of GPAW compensated constrained LTI system from an equivalent linear system with partial state constraints [Hou and Michel 1998]
  - GPAW controller can be defined by solution to a convex quadratic program or projection onto convex polyhedral cone problem [Teo and How 2010]
Conclusions

- Recalled GPAW scheme characteristics
- Main results:
  - controller state-output consistency
  - geometric bounding condition
- New results:
  - can infer stability for a class of GPAW compensated constrained LTI system from an equivalent linear system with partial state constraints [Hou and Michel 1998]
  - GPAW controller can be defined by solution to a convex quadratic program or projection onto convex polyhedral cone problem [Teo and How 2010]

Questions?


