Geometric Properties of Gradient Projection Anti-windup Compensated Systems

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Outline

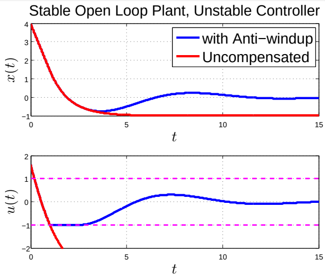
Motivation

Well Recognized Fact [Bernstein and Michel 1995]

Control saturation affects virtually all practical control systems.

Effects called windup [Kothare et al. 1994, Edwards and Postlethwaite 1998]. On control saturation,

- Performance degradation (with certainty)
- Instability (possibly)



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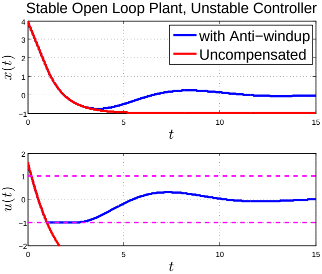

- Proposed Gradient Projection Anti-windup (GPAW) scheme [Teo and How 2009] for **nonlinear** MIMO systems/controllers, an **open problem** [Tarbouriech and Turner 2009]
- So far, stability results only for constrained planar LTI systems
- Here, prove state-output consistency and a geometric property

GPAW Scheme Overview

Saturated plant: $\dot{x} = f(x, \mathrm{sat}(u))$, $y = g(x, \mathrm{sat}(u))$. Nominal controller and GPAW compensated controller

$$\begin{aligned} \dot{x}_c &= f_c(x_c, y, r(t)) \\ u &= g_c(x_c, y, r(t)) \end{aligned} \quad \xrightarrow{\text{GPAW}, \Gamma = \Gamma^{\mathrm{T}} > 0} \quad \begin{aligned} \dot{x}_g &= R_{\mathcal{I}^*} f_c(x_g, y, r(t)) \\ u &= g_c(x_g, y, r(t)) \end{aligned}$$

Everything rests on $R_{\mathcal{I}^*}$!

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- $R_{\mathcal{I}^*}$ defined by **online** solution to a combinatorial optimization subproblem or **convex** quadratic program [Teo and How 2010]
- generalizes conditional integration method [Fertik and Ross 1967] using ideas from gradient projection method [Rosen 1960, 1961]
- attempts to maintain controller state-output consistency. So far, achieved: $\mathrm{sat}(u) \approx u$. Proven here: $\mathrm{sat}(u) = u$, as desired

GPAW Scheme Visualization

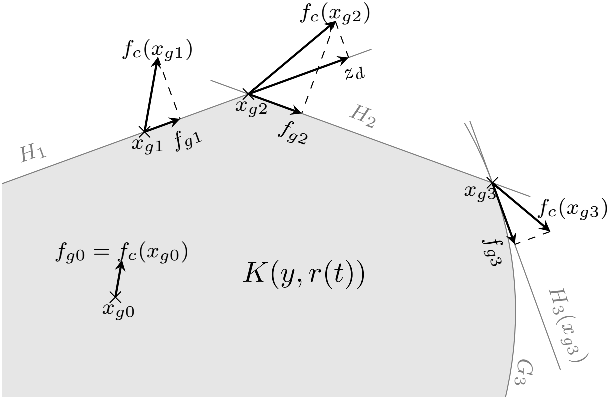
Nominal controller:

$$\dot{x}_c = f_c(x_c, y, r(t))$$
$$u = g_c(x_c, y, r(t))$$

GPAW controller:

$$\dot{x}_g = f_g(x_g, y, r(t))$$
$$u = g_c(x_g, y, r(t))$$

H_1, H_2, G_3 induced by **saturation**



- Note: $f_c(x_{gi}) := f_c(x_{gi}, y, r(t))$ and $f_{gi} := f_g(x_{gi}, y, r(t))$
- Unsaturated region is $K(y, r(t)) = \{x_g \in \mathbb{R}^q \mid \text{sat}(g_c(x_g, y, r(t))) = g_c(x_g, y, r(t))\}$

Controller State-output Consistency

- For output $u = g_c(x_g, y, r(t))$, get $\dot{u} = \frac{\partial g_c}{\partial x_g}\dot{x}_g + \frac{\partial g_c}{\partial y}\dot{y} + \frac{\partial g_c}{\partial r}\dot{r}$
- Previously identified limitation: when $\left\|\frac{\partial g_c}{\partial x_g}\dot{x}_g\right\| \ll \left\|\frac{\partial g_c}{\partial y}\dot{y} + \frac{\partial g_c}{\partial r}\dot{r}\right\|$, then GPAW scheme likely ineffective (only modify controller state)
- Restrict consideration to controllers with output equations **not depending** on $(y, r(t))$, ie.

$u = g_c(x_g)$ and **NOT** $u \neq g_c(x_g, y, r(t))$

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Theorem (Controller Output Consistency)

Consider the GPAW compensated controller whose output equation is of the form $u = g_c(x_g)$. If there exists a $T \in \mathbb{R}$ such that $\mathrm{sat}(u(T)) = u(T)$, then $\mathrm{sat}(u(t)) = u(t)$ holds for all $t \geq T$.

Unique property among anti-windup schemes (less specializations)

Implications

Without controller state-output consistency, closed loop system is

$$\begin{aligned} \dot{x} &= f(x, \mathrm{sat}(g_c(x_g))) \\ \dot{x}_g &= R_{\mathcal{I}^*} f_c(x_g, g(x, \mathrm{sat}(g_c(x_g))), r(t)) \end{aligned}$$

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- All complications arising from saturation accounted for by $R_{\mathcal{I}^*}$
- Allows stability conditions to be stated in terms of **unconstrained** dynamics

Approximate Nominal Controller

- Previously restricted to controllers with $u = g_c(x_c)$. Now consider when $u = g_c(x_c, y)$ (same construction when $u = g_c(x_c, y, r(t))$)
- "Design" unity DC gain, exponentially stable low-pass filter

$$\dot{\hat{y}} = a(y - \hat{y}) \qquad \hat{y}(0) = y(0)$$

and replace measurement y by its **approximation** only in the output

$$u = g_c(x_c, y) \qquad \Longrightarrow \qquad u = g_c(x_c, \hat{y})$$

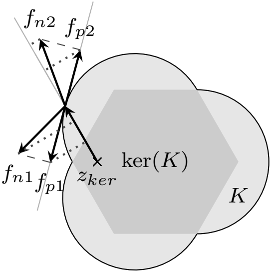
- Approximate nominal controller with augmented state $\tilde{x}_c := (x_c, \hat{y})$ is

$$\begin{aligned} \dot{x}_c &= f_c(x_c, y, r(t)) \\ u &= g_c(x_c, y) \end{aligned} \qquad \underset{\approx}{\xrightarrow{a \gg 1}} \qquad \begin{aligned} \dot{x}_c &= f_c(x_c, y, r(t)) \\ \dot{\hat{y}} &= a(y - \hat{y}) \\ u &= g_c(x_c, \hat{y}) = g_c(\tilde{x}_c) \end{aligned}$$

- Singular perturbation theory [Khalil 2002] shows that approximation can be made arbitrarily well as $a \to \infty$

Geometric Bounding Condition

- Let K be unsaturated region, $K = \{x \in \mathbb{R}^q \mid \text{sat}(g_c(x)) = g_c(x)\}$
- Let $f_n(t,z)$, $f_p(t,z)$ be the vector fields of **uncompensated** ($\dot{z} = f_n(t,z)$) and GPAW compensated systems ($\dot{z} = f_p(t,z)$), Γ the GPAW parameter



Theorem (Geometric Bounding Condition)

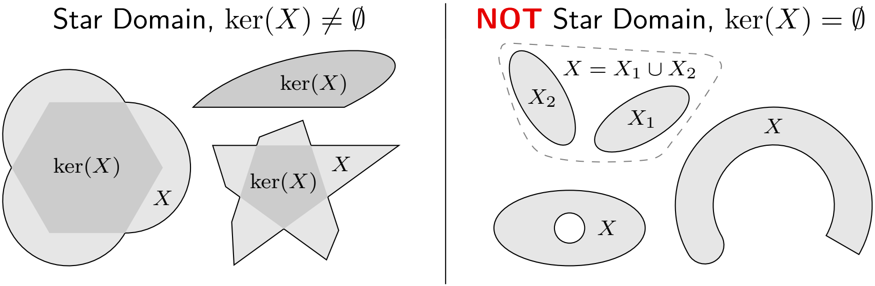
If unsaturated region $K \subset \mathbb{R}^q$ is a star domain, then for any $z \in (\mathbb{R}^n \times K)$ and any $z_{ker} \in (\mathbb{R}^n \times \ker(K))$, the geometric condition

$$\langle z - z_{ker}, \tilde{\Gamma}^{-1} f_p(t,z)\rangle \leq \langle z - z_{ker}, \tilde{\Gamma}^{-1} f_n(t,z)\rangle,$$

holds for all $t \in \mathbb{R}$, where $\tilde{\Gamma} = \left[\begin{smallmatrix} I & 0 \\ 0 & \Gamma \end{smallmatrix}\right]$

Star Domains

Examples and counterexamples of Star Domains in $\mathbb{R}^2$:



- Any convex set X is also a star domain with $\ker(X) = X$
- For any non-convex star domain, $\ker(X)$ is a strict subset of X
- If X is a star domain, then $\mathbb{R}^n \times X$ is also a star domain with kernel $\mathbb{R}^n \times \ker(X)$

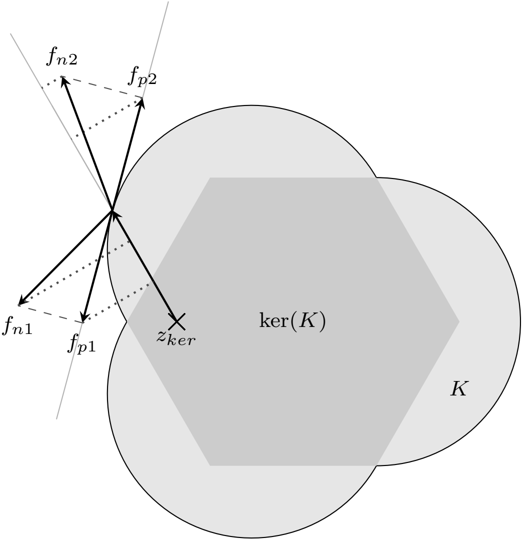
Geometric Interpretation

Closed-loop systems:
Uncompensated,

$$\dot{z} = f_n(t, z)$$

GPAW compensated,

$$\dot{z} = f_p(t, z)$$



Implications

Geometric Implications

If a Lyapunov function $V(x, x_c)$ exists for the **uncompensated** system such that on the boundary of the unsaturated region $\mathbb{R}^n \times K$, $\frac{\partial V}{\partial x_c}$ always points **out** from the kernel $\ker(K)$, then it is also a Lyapunov function for the GPAW compensated system!

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Conjecture

That for any Lyapunov function for any uncompensated system, there exists a derived Lyapunov function satisfying above property

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Proof of geometric condition in ACC paper is faulty. Corrected in:

J. Teo and J. P. How. Corrections to 'geometric properties of gradient projection anti-windup compensated systems'. Technical Report ACL10-02, MIT, Cambridge, MA, June 2010. URL http://hdl.handle.net/1721.1/56001. Aerosp. Controls Lab.

Conclusions

- Recalled GPAW scheme characteristics
- Main results:
  - controller state-output consistency
  - geometric bounding condition
- New results:
  - can infer stability for a class of GPAW compensated constrained LTI system from an equivalent linear system with partial state constraints [Hou and Michel 1998]
  - GPAW controller can be defined by solution to a convex quadratic program or projection onto convex polyhedral cone problem [Teo and How 2010]

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Questions?

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