Geometric Properties of Gradient Projection Anti-windup Compensated Systems

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***Abstract*— The *gradient projection anti-windup* (GPAW) scheme was recently proposed as an anti-windup method for *nonlinear* multi-input-multi-output systems/controllers, which was recognized as a largely open problem in a recent survey paper. Here, we show that for controllers whose output equation depends only on its state, the GPAW compensated controller achieves *exact* state-output consistency when appropriately initialized. In a related paper analyzing the GPAW scheme on a simple constrained system, this property was crucial in proving that the GPAW scheme can only maintain/enlarge the *exact* region of attraction of the uncompensated system. When the nominal controller does not have the required structure, an arbitrarily close approximating controller can be constructed. Further geometric properties of GPAW compensated systems are then presented, which illuminates the role of the GPAW tuning parameter.**

I. Introduction

The *gradient projection anti-windup* (GPAW) scheme was proposed in [1] as an anti-windup method for *nonlinear* multi-input-multi-output (MIMO) systems/controllers. It was recognized in a recent survey paper [2] that anti-windup compensation for nonlinear systems remains largely an *open problem*. To this end, [3] and relevant references in [2] represent some recent advances. The GPAW scheme uses a continuous-time extension of the gradient projection method of nonlinear programming [4], [5] to extend the "stop integration" heuristic outlined in [6] to the case of nonlinear MIMO systems/controllers. Application of the GPAW scheme to some nominal controllers results in a *hybrid* GPAW compensated controller [1], and hence a hybrid closed loop system.

In a related paper [7], we analyzed the GPAW scheme when applied to a constrained first order linear time invariant (LTI) system driven by a first order LTI controller, where the objective is to regulate the system state about the origin. The main result of [7] shows that the GPAW scheme can only maintain/enlarge the *exact* region of attraction of the uncompensated system. This shows the GPAW scheme to be a valid anti-windup method for this simple system. A crucial part of these results relies on the fact that the GPAW compensated controller maintains *exact* state-output consistency, ie. $\mathrm{sat}(u) \equiv u$, when the controller state is appropriately initialized. While this fact is easily seen for the simple system considered in [7], it is not immediately clear for a more general GPAW compensated controller. We present this result for general GPAW compensated controllers as described below.

We first describe the nominal tracking system and the derived GPAW compensated controller in Sections II and III respectively. It is shown in Theorem 1 that when appropriately initialized, the GPAW compensated controller achieves *exact* state-output consistency, ie. $\mathrm{sat}(u) \equiv u$ for all times, provided that the output equation of the nominal controller depends only on its state, and specifically, not on measurements or exogenous signals. When the nominal controller does not possess this property, we show in Section IV how an arbitrarily close approximating controller can be constructed. In Section V, we present further geometric properties of GPAW compensated systems, the main result of which, Theorem 2, illuminates the role of the GPAW tuning parameter.

We will adopt the following conventions in the sequel. For inequalities involving vectors, the inequality is to be interpreted element-wise. Let $\mathcal{I}$ and $\mathcal{J}$ be two sets. The cardinality of $\mathcal{I}$ will be denoted by $|\mathcal{I}|$, and $\mathcal{I} \setminus \mathcal{J}$ is the relative complement of $\mathcal{J}$ in $\mathcal{I}$. For any set $X \in \mathbb{R}^n$, the boundary of X is denoted by ∂X. The dot product of two vectors $x, y \in \mathbb{R}^n$ is denoted by $\langle x, y \rangle \in \mathbb{R}$.

II. Constrained Nominal System

Consider the input constrained nonlinear system

$$\begin{aligned} \dot{x} &= f(x, \mathrm{sat}(u)), & x(0) &= x_0, \\ y &= g(x, \mathrm{sat}(u)), \end{aligned}$$

where $x, x_0 \in \mathbb{R}^n$ are the state and initial state, $u \in \mathbb{R}^m$ is the plant input, $y \in \mathbb{R}^p$ is the measurement, and $\mathrm{sat}\colon \mathbb{R}^m \to \mathbb{R}^m$ is the familiar saturation function. Let $C^k([0,\infty),\mathbb{R}^n)$ be the vector space of k times continuously differentiable functions $[0,\infty) \to \mathbb{R}^n$, and let $\mathcal{R} \subset C^1([0,\infty),\mathbb{R}^n)$ be a class of admissible reference signals evolving in $\mathbb{R}^n$ that is at least continuously differentiable. Assume that the control objective is to have x track a reference signal $r \in \mathcal{R}$, so that the *instantaneous* tracking error at time t is $e = x - r(t)$. The *time-varying* tracking error dynamics are then given by

$$\begin{aligned} \dot{e} &= f(e + r(t), \mathrm{sat}(u)) - \dot{r}(t), \quad e(0) = x_0 - r(0), \\ y &= g(e + r(t), \mathrm{sat}(u)). \end{aligned} \tag{1}$$

For control designs that require smoother than C^1 reference signals, we can always restrict $\mathcal{R}$ appropriately, eg. by requiring that $\mathcal{R} \subset C^k([0,\infty),\mathbb{R}^n)$, so that we can define the *instantaneous* controller reference $\tilde{r}(t) =$

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$(r(t), \dot{r}(t), \dots, r^{(k)}(t)) \in \mathbb{R}^{(k+1)n}$, where $r^{(i)}(t)$ denotes the i-th time derivative of r at time t. Let the *nominal* controller take the form

$$\begin{aligned} \dot{x}_c &= f_c(x_c, y, \tilde{r}(t)), \quad x_c(0) = x_{c0}, \\ u &= g_c(x_c), \end{aligned} \tag{2}$$

where $x_c, x_{c0} \in \mathbb{R}^q$ are the state and initial state, $u \in \mathbb{R}^m$ is the output, and g_c *depends only on the controller state* x_c.

Remark 1: Memoryless static feedback controllers of the form $u = g_c(y, \tilde{r}(t))$ can be approximated to have the form of (2) (see Section IV). □

Remark 2: It is assumed in (2) that $r(t)$ and all its time derivatives up to the k-th order are available to the controller. See [8, pp. 194–195] for a justification. □

The closed loop system defined by (1) and (2) is called the *nominal system*, which can be written as

$$\begin{aligned} \dot{e} &= f(e + r(t), \mathrm{sat}(g_c(x_c))) - \dot{r}(t), \\ \dot{x}_c &= f_c(x_c, g(e + r(t), \mathrm{sat}(g_c(x_c))), \tilde{r}(t)), \end{aligned} \tag{3}$$

with initial state $(e(0), x_c(0)) = (x_0 - r(0), x_{c0})$.

III. Gradient Projection Anti-windup Compensated System

Here, we apply the GPAW scheme [1] on (2) and show that the GPAW compensated controller achieves *exact state-output consistency*, ie. $\mathrm{sat}(u) \equiv u$, for "almost all" times (stated more precisely as Theorem 1) when g_c depends only on the controller state. The GPAW compensated controller is derived from (2) and takes the form

$$\begin{aligned} \dot{x}_g &= f_g(x_g, y, \tilde{r}(t)), \quad x_g(0) = x_{c0}, \\ u &= g_c(x_g), \end{aligned} \tag{4}$$

in which the only difference with (2) is the definition of an independent state $x_g \in \mathbb{R}^q$ and the controller state update law f_g. The following shows how f_g is constructed [1], leading to its definition in (8).

Remark 3: Even though the GPAW controller (4) may not appear to conform to the conventional anti-windup paradigm where the nominal controller is to remain unaltered, it can always be transformed in a way such that the nominal controller need not be modified. For example, if the anti-windup compensator's output is to be combined additively with that of the nominal controller, and the output of the nominal controller can be measured, then by subtracting the nominal controller's output from that of the GPAW controller's, we obtain the desired anti-windup signal. Alternatively, one can build a model of the nominal controller, and with knowledge of the initial controller state, the same can be achieved. We avoid the difficulties associated with the individual robustness issues of each *realization* by focusing only on the effective *composite* controller (4). □

In the following, we consider a fixed point in time, so that $(x_g, y, \tilde{r}(t)) \in \mathbb{R}^{q+p+(k+1)n}$ are fixed. Let $\mathcal{I}_j = \{1, 2, \dots, j\}$ where j is some positive integer. First, observe that the saturation function $\mathrm{sat}\colon \mathbb{R}^m \to \mathbb{R}^m$ in (1) is defined with m lower and upper saturation limits $u_{iL}, u_{iU} \in \mathbb{R}$ satisfying $u_{iL} < u_{iU}$ for $i \in \mathcal{I}_m$. Let g_c in (4) be decomposed as

$$g_c(x_g) = [g_{c1}(x_g), g_{c2}(x_g), \dots, g_{cm}(x_g)]^{\mathrm{T}}.$$

The GPAW scheme constructs f_g in a way to maintain the feasibility of the $2m$ saturation constraints

$$\begin{aligned} h_i(x_g) &= g_{ci}(x_g) - u_{iU} \leq 0, \\ h_{i+m}(x_g) &= -g_{ci}(x_g) + u_{iL} \leq 0, \end{aligned} \tag{5}$$

with associated gradient vectors

$$\nabla h_i(x_g) = -\nabla h_{i+m}(x_g) = \left(\frac{\partial g_{ci}}{\partial x_c}(x_g) \right)^{\mathrm{T}} \in \mathbb{R}^q,$$

for $i \in \mathcal{I}_m$. For any non-empty set of indices $\mathcal{I} \subset \mathcal{I}_{2m}$, $|\mathcal{I}| = s > 0$, define the $q \times s$ matrix

$$N_{\mathcal{I}}(x_g) = [\nabla h_{\sigma_{\mathcal{I}}(1)}(x_g), \nabla h_{\sigma_{\mathcal{I}}(2)}(x_g), \dots, \nabla h_{\sigma_{\mathcal{I}}(s)}(x_g)],$$

where $\sigma_{\mathcal{I}}\colon \mathcal{I}_s \to \mathcal{I}$ is *any* chosen bijection that assigns an integer in $\mathcal{I}$ to an integer in $\mathcal{I}_s = \{1, 2, \dots, s\}$. For $\mathcal{I} = \emptyset$, define $N_{\mathcal{I}}(x_g) = 0 \in \mathbb{R}^q$.

Remark 4: Any bijection $\sigma_{\mathcal{I}}\colon \mathcal{I}_s \to \mathcal{I}$ suffices. For example, we can take the ascending order map defined recursively by $\sigma_{\mathcal{I}}(i) = \min\left(\mathcal{I} \setminus \cup_{j=1}^{i-1}\{\sigma_{\mathcal{I}}(j)\}\right)$ for all $i \in \mathcal{I}_s$. The final result will be independent of the choice of $\sigma_{\mathcal{I}}$. □

In contrast to numerous anti-windup schemes, the GPAW scheme has only a *single* tuning parameter, a chosen symmetric positive definite matrix $\Gamma \in \mathbb{R}^{q \times q}$. For any $\mathcal{I} \subset \mathcal{I}_{2m}$ such that $|\mathcal{I}| = 0$, or $0 < |\mathcal{I}| <= q$ and $N_{\mathcal{I}}(x_g)$ is full rank, define

$$f_{\mathcal{I}}(x_g, y, \tilde{r}(t)) = R_{\mathcal{I}}(x_g) f_c(x_g, y, \tilde{r}(t)), \tag{6}$$

where

$$R_{\mathcal{I}}(x_g) = \begin{cases} I - \Gamma N_{\mathcal{I}} \left(N_{\mathcal{I}}^{\mathrm{T}} \Gamma N_{\mathcal{I}} \right)^{-1} N_{\mathcal{I}}^{\mathrm{T}}(x_g), & \text{if } |\mathcal{I}| > 0, \\ I, & \text{otherwise.} \end{cases}$$

Define the set of indices corresponding to active saturation constraints as

$$\mathcal{I}_{sat} = \{ i \in \mathcal{I}_{2m} \mid h_i(x_g) \geq 0 \}.$$

Let $\mathcal{J}$ be the set of all subsets of $\mathcal{I}_{sat}$ with cardinality less than or equal to q. Define the following *combinatorial* optimization subproblem

$$\begin{aligned} \max_{\mathcal{I} \in \mathcal{J}} \; & f_c^{\mathrm{T}}(x_g, y, \tilde{r}(t)) \Gamma^{-1} f_{\mathcal{I}}(x_g, y, \tilde{r}(t)), \\ \text{subject to} \quad & \mathrm{rank}(N_{\mathcal{I}}(x_g)) = |\mathcal{I}|, \\ & N_{\mathcal{I}_{sat} \setminus \mathcal{I}}^{\mathrm{T}}(x_g) f_{\mathcal{I}}(x_g, y, \tilde{r}(t)) \leq 0. \end{aligned} \tag{7}$$

The following result asserts the existence of solutions to subproblem (7).

Proposition 1: For any fixed $(x_g, y, \tilde{r}(t)) \in \mathbb{R}^{q+p+(k+1)n}$, there exists a solution to subproblem (7).

Proof: To simplify the notation, we will omit all function arguments. If $\mathcal{I}_{sat} = \emptyset$, then $\mathcal{J} = \{\emptyset\}$, and it can be verified that $\mathcal{I}^* = \emptyset$ is the unique optimal solution. If $\mathrm{rank}(N_{\mathcal{I}_{sat}}) = v$ (necessarily $\leq q$), let $\mathcal{I}$ ($\subset \mathcal{I}_{sat}$) be any set of indices of v *linearly independent* gradient vectors, ∇h_i for $i \in \mathcal{I}_{sat}$, so that $\mathrm{rank}(N_{\mathcal{I}}) = v = |\mathcal{I}|$. Since $\mathrm{rank}(N_{\mathcal{I}_{sat}}) = v$, the columns of $N_{\mathcal{I}_{sat}}$ are *linearly* *dependent* if $s := |\mathcal{I}_{sat}| > v$. Then, $N_{\mathcal{I}_{sat}\setminus\mathcal{I}} \in \mathbb{R}^{q\times(s-v)}$ can be written as $N_{\mathcal{I}_{sat}\setminus\mathcal{I}} = N_{\mathcal{I}}\Psi$ for some $\Psi \in \mathbb{R}^{v\times(s-v)}$. It can be verified from (6) that $N^{\mathrm{T}}_{\mathcal{I}_{sat}\setminus\mathcal{I}} f_{\mathcal{I}} = \Psi^{\mathrm{T}} N^{\mathrm{T}}_{\mathcal{I}} f_{\mathcal{I}} = 0 \in \mathbb{R}^{s-v}$, and hence $\mathcal{I}$ is a feasible (not necessarily optimal) solution to subproblem (7). Since there can only be a finite number of active saturation constraints, $|\mathcal{I}_{sat}| = s < \infty$, the number of candidate solutions ($\sum_{i=0}^{\min\{q,s\}} \binom{s}{i}$) is also finite. It follows that optimal solutions always exist that can be found by an exhaustive search algorithm. ■

At each fixed time (so that $(x_g, y, \tilde{r}(t))$ is fixed), let $\mathcal{I}^*$ be a solution to subproblem (7). The GPAW compensated controller derived from (2) is then given by (4) with

$$f_g(x_g, y, \tilde{r}(t)) = f_{\mathcal{I}^*}(x_g, y, \tilde{r}(t)). \tag{8}$$

The following is a key property of general GPAW compensated controllers that was crucial in obtaining the results of [7]. For the particular first order controller in [7], this property is readily seen by inspecting the defining equations of the GPAW compensated controller. However, this property is not immediately clear for more general GPAW compensated controllers, and it is shown here.

Theorem 1 (Controller State-Output Consistency): Consider the GPAW compensated controller defined by (4) and (8). If there exists a $T \in \mathbb{R}$ such that $\mathrm{sat}(u(T)) = u(T)$, then $\mathrm{sat}(u(t)) = u(t)$ holds for all $t \geq T$.

Proof: Observe that $\mathrm{sat}(u(t)) = u(t)$ if and only if $h_i(x_g(t)) \leq 0$ for all $i \in \mathcal{I}_{2m}$ (see (5)). By assumption, we have $h_i(x_g(T)) \leq 0$, for all $i \in \mathcal{I}_{2m}$. Hence it is sufficient to show that for all $i \in \mathcal{I}_{2m}$, whenever $h_i(x_g(t)) = 0$, then $\dot{h}_i(x_g(t)) \leq 0$ holds. Taking the time derivative, we have

$$\dot{h}_i(x_g(t)) = \nabla h_i^{\mathrm{T}}(x_g(t)) f_{\mathcal{I}^*}(x_g(t), y(t), \tilde{r}(t)).$$

If $h_i(x_g(t)) = 0$, then $i \in \mathcal{I}_{sat}$. We need to show that

$$N^{\mathrm{T}}_{\mathcal{I}_{sat}}(x_g(t)) f_{\mathcal{I}^*}(x_g(t), y(t), \tilde{r}(t)) \leq 0, \tag{9}$$

or equivalently, since $\mathcal{I}^* \subset \mathcal{I}_{sat}$,

$$N^{\mathrm{T}}_{\mathcal{I}_{sat}\setminus\mathcal{I}^*}(x_g(t)) f_{\mathcal{I}^*}(x_g(t), y(t), \tilde{r}(t)) \leq 0,$$
$$N^{\mathrm{T}}_{\mathcal{I}^*}(x_g(t)) f_{\mathcal{I}^*}(x_g(t), y(t), \tilde{r}(t)) \leq 0, \quad \text{if } \mathcal{I}^* \neq \emptyset.$$

Observe that the second inequality above needs to be satisfied only when $\mathcal{I}^* \neq \emptyset$ since for $\mathcal{I}^* = \emptyset$, the first inequality is equivalent to (9). Since $\mathcal{I}^*$ is a solution to subproblem (7), the first inequality holds due to the second constraint of subproblem (7). For $\mathcal{I}^* \neq \emptyset$, the definition of $f_{\mathcal{I}^*}$ (6) yields

$$N^{\mathrm{T}}_{\mathcal{I}^*}(x_g(t)) f_{\mathcal{I}^*}(x_g(t), y(t), \tilde{r}(t)) = 0 \in \mathbb{R}^{|\mathcal{I}^*|},$$

so that the second inequality holds. Since these two inequalities hold for all $t \in \mathbb{R}$, the conclusion follows. ■

Remark 5: Note that Theorem 1 depends critically on h_i (and hence g_c) being dependent only on the controller state. See [7, Fig. 3] for a numerical illustration of this result. □

The closed loop system defined by (1), (4) and (8) is called the *GPAW compensated system*, rewritten as

$$\begin{aligned} \dot{e} &= f(e + r(t), \mathrm{sat}(g_c(x_g))) - \dot{r}(t), \\ \dot{x}_g &= f_{\mathcal{I}^*}(x_g, g(e + r(t), \mathrm{sat}(g_c(x_g))), \tilde{r}(t)), \end{aligned} \tag{10}$$

with initial state $(e(0), x_g(0)) = (x_0 - r(0), x_{c0})$. Observe that if there exists a $T \in \mathbb{R}$ (possibly $T = 0$) such that $\mathrm{sat}(g_c(x_g(T))) = g_c(x_g(T))$, then for all $t \geq T$, Theorem 1 allows (10) to be simplified to

$$\begin{aligned} \dot{e} &= f(e + r(t), g_c(x_g)) - \dot{r}(t), \\ \dot{x}_g &= f_{\mathcal{I}^*}(x_g, g(e + r(t), g_c(x_g)), \tilde{r}(t)). \end{aligned}$$

IV. Approximating Controller

Theorem 1 requires that the output equation of the nominal controller (2) depends only on the controller state. If this is not true, we show here that an arbitrarily close approximation of the nominal controller can be constructed that has the required structure. Note that this construction is not unique. The main idea is to replace the signal components in the controller output equation that are not part of the controller state by its low-pass filtered signal, and design the low-pass filter such that its bandwidth is much larger than the effective bandwidth of the closed loop system. It is clear that the approximation will be enhanced as the bandwidth of the low-pass filter is increased. Importantly, the main purpose of this low-pass filter is *not* for noise rejection or performance/robustness enhancements.

Consider the nominal controller

$$\begin{aligned} \dot{x}_c &= f_c(x_c, y, \tilde{r}(t)), \quad x_c(0) = x_{c0}, \\ u &= g_c(x_c, y), \end{aligned} \tag{11}$$

whose output equation depends not only on the state, but on measurement y as well. For simplicity, we have assumed that the output equation is not dependent on the controller reference $\tilde{r}(t)$. If it indeed does, the treatment is similar, and also simpler due to the structure of $\tilde{r}(t)$.

Remark 6: When g_c depends on the measurement y as in (11), the closed loop system (1), (11) will contain an *algebraic loop* whenever $\frac{\partial g}{\partial u}\frac{\partial g_c}{\partial y} \not\equiv 0$. □

Consider augmenting the controller state to be $\tilde{x}_c = (x_c, \tilde{y})$, with $\tilde{y} = y$. Then the controller output equation $u = g_c(x_c, \tilde{y}) = g_c(\tilde{x}_c)$ will be of the desired form (2). The state equation of the augmented controller with state $\tilde{x}_c$ needs to satisfy

$$\dot{x}_c = f_c(x_c, y, \tilde{r}(t)), \qquad \dot{\tilde{y}} = \dot{y}. \tag{12}$$

Clearly, if the functions f and g in (1) are known *exactly*, realization of (12) is straightforward[1] by taking the time derivative of y in (1) and using the knowledge of f and g. We avoid making such a conservative assumption by using an *approximation*. Consider $\tilde{y}$ obtained as the output of an *exponentially stable, unity DC gain* low-pass filter with input y, parameterized by $a \in (0, \infty)$

$$\dot{\tilde{y}} = a(y - \tilde{y}), \qquad \tilde{y}(0) = y(0).$$

It can be seen that $\tilde{y}(t) \to y(t)$ for all $t \geq 0$ as $a \to \infty$, so that the solution of the approximating controller can be made arbitrarily close to the nominal controller. While this can be shown formally for any fixed $y(t)$, $t \in [0, \infty)$ and $r \in \mathcal{R}$

[1] When the closed loop system contains an algebraic loop, there are additional difficulties on well-posedness of the feedback interconnection.

using singular perturbation theory [9, Chapter 11, pp. 423 – 468], the larger question is the effect of the approximation on the *closed loop system*, which we discuss next.

The approximate controller by the above considerations is

$$\begin{aligned} \dot{x}_c &= f_c(x_c, y, \tilde{r}(t)), & x_c(0) &= x_{c0}, \\ \dot{\tilde{y}} &= a(y - \tilde{y}), & \tilde{y}(0) &= y(0), \\ u &= g_c(x_c, \tilde{y}), \end{aligned}$$

which, together with (1), gives the closed loop dynamics

$$\begin{aligned} \dot{e} &= f(e + r(t), \mathrm{sat}(g_c(x_c, \tilde{y}))) - \dot{r}(t), \\ \dot{x}_c &= f_c(x_c, g(e + r(t), \mathrm{sat}(g_c(x_c, \tilde{y}))), \tilde{r}(t)), \\ \epsilon\dot{\tilde{y}} &= g(e + r(t), \mathrm{sat}(g_c(x_c, \tilde{y}))) - \tilde{y}, \end{aligned} \tag{13}$$

where $\epsilon = \frac{1}{a}$. Observe that when $\epsilon = 0$, we recover the *exact* closed loop system obtained with controller (11), which corresponds to the reduced system in the singular perturbation framework. Here, we refer to (13) as the approximate system, and (13) with $\epsilon = 0$ as the exact system. When we assume existence and uniqueness of solutions[2] to the *exact* system, then (13) is a standard singular perturbation model [9, pp. 424]. It can be shown that if g and g_c are such that the eigenvalue condition [9, pp. 433]

$$\mathrm{Re}\left(\lambda\left(\frac{\partial g}{\partial u}\frac{\partial g_c}{\partial y} - I\right)\right) < 0,$$

holds uniformly for all (t, e, x_c) in some domain, then the origin of the associated boundary layer model for the singular perturbation model (13) is exponentially stable. With this, and assuming existence and uniqueness of solutions of the exact system, [9, Theorem 11.1, pp. 434] shows that on any finite time interval, the solution of the approximate system can be made arbitrarily close to the solution of the exact system when ϵ is sufficiently small (a is sufficiently large). When the origin is an *exponentially stable* equilibrium of the *exact system*, [9, Theorem 11.2, pp. 439 – 440] shows that the result extends to infinite intervals.

Observe that for constrained LTI systems driven by LTI controllers, local exponential stability is usually guaranteed so that the infinite time approximation result holds. If the exact system is not exponentially stable and the finite time approximation result indicated above is not sufficient, re-doing the analysis with the approximate controller may be required. Because the approximation can be made arbitrarily well, it is likely that the approximate controller will be able to achieve the control objectives as well.

V. Further Geometric Properties of GPAW Compensated Systems

This section presents further geometric properties of GPAW compensated systems, the main result of which (Theorem 2) illuminates the role of the GPAW tuning parameter, Γ. These geometric properties are foreseen to be needed to extend the results of [7] and to prove general desirable properties of GPAW compensated systems.

First, we describe star domains, which will be needed to describe the unsaturated regions for GPAW compensated systems. For any two points $x_1, x_2 \in \mathbb{R}^n$, let the line segment connecting them be

$$\eta(x_1, x_2) = \{x \in \mathbb{R}^n \mid x = \theta x_1 + (1-\theta)x_2, \forall\theta \in [0,1]\}.$$

Definition 1: [10, Definition 1.4, pp. 5] Let $X \subset \mathbb{R}^n$ be a nonempty set. The *kernel* of X, denoted by $\ker(X)$, is

$$\ker(X) = \{x \in \mathbb{R}^n \mid \eta(x, y) \subset X, \forall y \in X\} \subset X.$$

Definition 2: [10, Definition 1.2, pp. 4] A nonempty set $X \subset \mathbb{R}^n$ is a *star domain*, or star-shaped, if $\ker(X) \neq \emptyset$. In other words, a nonempty set X is a star domain if there exists at least one point $x \in X$ such that for every $y \in X$, the line segment connecting x and y is contained within X.

Remark 7: Clearly, any convex set X is also a star domain with $\ker(X) = X$. For any non-convex star domain, $\ker(X)$ is a strict subset of X. □

Remark 8: If $X \subset \mathbb{R}^n$ is a star domain, then $X \times \mathbb{R}^m$ is also a star domain in $\mathbb{R}^{n+m}$ with kernel $\ker(X) \times \mathbb{R}^m$. □

When a star domain is defined by a set of constraint functions, the following gives a characterization of the gradient vectors of the constraint functions on the boundary of the star domain.

Lemma 1: Let X be defined by a set of m constraints

$$X = \{x \in \mathbb{R}^n \mid \tilde{h}_i(x) \leq 0, \forall i \in \mathcal{I}_m\} \subset \mathbb{R}^n.$$

For any boundary point $x \in \partial X$, define

$$\mathcal{I}_{lim}(x) = \{i \in \mathcal{I}_m \mid \tilde{h}_i(x) = 0\}.$$

If X is a star domain, then for any $x_{ker} \in \ker(X)$ and any boundary point $x \in \partial X$, we have

$$\langle x - x_{ker}, \nabla\tilde{h}_i(x)\rangle \geq 0, \qquad \forall i \in \mathcal{I}_{lim}(x).$$

Proof: By the definition of $\ker(X)$ and the star domain, we have $y(\theta) := \theta x + (1-\theta)x_{ker} \in X$ for all $\theta \in [0,1]$. Hence $\tilde{h}_i(y(\theta)) \leq 0$ for all $i \in \mathcal{I}_m$. View $\tilde{h}_i(y(\theta))$ as a function of θ with x, x_{ker} fixed. Since $x \in \partial X$, we must have $\tilde{h}_i(y(1)) = \tilde{h}_i(x) = 0$ for all $i \in \mathcal{I}_{lim}(x)$. Since $\tilde{h}_i(y(\theta)) \leq 0$ for all $\theta \in [0,1]$, $\tilde{h}_i(y(\theta))$ must be non-decreasing at $\theta = 1$. Hence by the chain rule,

$$\frac{d\tilde{h}_i}{d\theta}(y(\theta)) = \frac{d\tilde{h}_i}{dx}(y(\theta))(x - x_{ker}) \geq 0, \quad \forall i \in \mathcal{I}_{lim}(x),$$

at $\theta = 1$. This gives $\frac{d\tilde{h}_i}{dx}(x)(x - x_{ker}) \geq 0$ for all $i \in \mathcal{I}_{lim}(x)$, which can be written in dot product form with the gradient vector as stated. ■

Let the nominal system (3) and GPAW compensated system (10) be represented as $\dot{z} = f_n(t, z)$ and $\dot{z} = f_p(t, z)$ respectively. Define the unsaturated region

$$K = \{x \in \mathbb{R}^q \mid h_i(x) \leq 0, \forall i \in \mathcal{I}_{2m}\} \subset \mathbb{R}^q,$$

where h_i are the saturation constraint functions in (5).

Remark 9: If g_c in (2) is a *linear* function, ie. $g_c(x) = C_c x$ where $C_c \in \mathbb{R}^{m\times q}$, then K is convex (in fact, a convex polyhedron), and hence is also a star domain in $\mathbb{R}^q$. □

[2] Recall that in the anti-windup context, the nominal controller has been designed to achieve some desired performance. Existence and uniqueness of solutions to the closed loop system is usually guaranteed even when not explicitly sought in the control design.

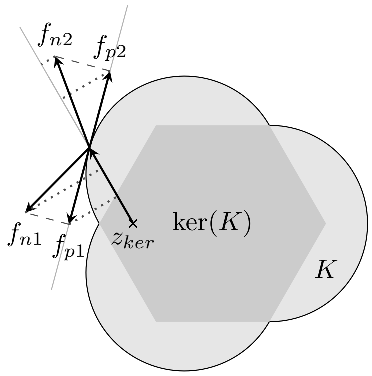

Fig. 1. Illustration of the geometric condition in Theorem 2 with an example non-convex star domain K. For $\Gamma = I$ and any $z \in \mathbb{R}^n \times \partial K$, $z_{ker} \in \mathbb{R}^n \times \ker(K)$, the projection of f_p onto $z - z_{ker}$ is always more negative (or less positive) than the projection of f_n onto the same vector.

The following is the main result of this section which illuminates a geometric property of all GPAW compensated systems.

Theorem 2 ([11, Theorem C1]): If $K \subset \mathbb{R}^q$ is a star domain, then for any $z \in (\mathbb{R}^n \times K)$ and any $z_{ker} \in (\mathbb{R}^n \times \ker(K))$,

$$\langle z - z_{ker}, \tilde{\Gamma}^{-1} f_p(t,z)\rangle \leq \langle z - z_{ker}, \tilde{\Gamma}^{-1} f_n(t,z)\rangle,$$

holds for all $t \in \mathbb{R}$, where $\tilde{\Gamma} = \left[\begin{smallmatrix}\Theta & 0\\ 0 & \Gamma\end{smallmatrix}\right] \in \mathbb{R}^{(n+q)\times(n+q)}$ and $\Theta \in \mathbb{R}^{n\times n}$ is any nonsingular square $n \times n$ matrix.

Proof: See [11]. ■

The geometric condition stated in Theorem 2 is illustrated in Fig. 1 with an example non-convex star domain for the unsaturated region. If the control objective is to regulate the system state about the origin, we can set $z_{ker} = 0$, provided $\ker(K)$ contains the origin of $\mathbb{R}^q$.

VI. Conclusions

When the output equation of the nominal controller depends only on the controller state, then exact controller state-output consistency is achieved for the GPAW compensated controller when appropriately initialized. When the nominal controller does not possess this structure, an arbitrarily close approximating controller can be constructed that has the required structure. Further geometric properties of GPAW compensated systems are established, which illuminates the role of the GPAW tuning parameter.

Note that the proof of Theorem 2 as originally presented in [12] was faulty. The present paper is a version of [12] incorporating corrections described in the technical report [11].

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