Analysis of Gradient Projection Anti-windup Scheme

What the Simplest Feedback System Reveals

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Outline

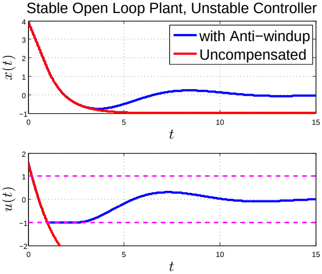
Motivation

Well Recognized Fact [Bernstein and Michel 1995]

Control saturation affects virtually all practical control systems.

Effects called windup [Kothare et al. 1994, Edwards and Postlethwaite 1998]. On control saturation,

- Performance degradation (with certainty)
- Instability (possibly)



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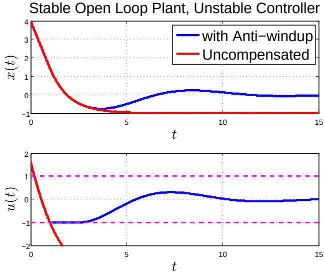

- Proposed Gradient Projection Anti-windup (GPAW) scheme [Teo and How 2009] for **nonlinear** MIMO systems/controllers, an **open problem** [Tarbouriech and Turner 2009]
- No prior stability results. First results here
- Can we say anything with simplest possible feedback system?

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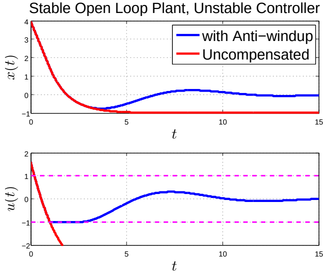

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- No prior stability results. First results here
- Can we say anything with simplest possible feedback system? A lot!

Simplest Possible Feedback System

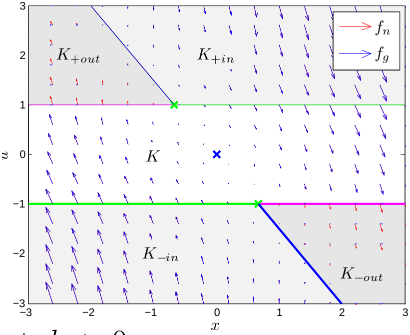
1-st order saturated LTI plant and 1-st order LTI controller:

$$\dot{x} = ax + b\,\mathrm{sat}(u) \qquad \dot{x}_c = \tilde{c}x_c + \tilde{d}x$$

$$\mathrm{sat}(u) = \max\{\min\{u, u_{max}\}, u_{min}\} \qquad u = \tilde{e}x_c$$

Nominal system transforms to:

$$\Sigma_n : \begin{cases} \dot{x} = ax + b\,\mathrm{sat}(u) \\ \dot{u} = cx + du \end{cases} \iff \dot{z} = f_n(z)$$



GPAW compensated **switched** system:

$$\Sigma_g : \begin{cases} \dot{x} = ax + b\,\mathrm{sat}(u) \\ \dot{u} = \begin{cases} 0, & \text{if } u \geq u_{max}, cx + du > 0 \\ 0, & \text{if } u \leq u_{min}, cx + du < 0 \\ cx + du, & \text{otherwise} \end{cases} \end{cases} \iff \dot{z} = f_g(z)$$

GPAW Compensated System as a Projected Dynamical System

- GPAW vector field f_g **discontinuous**
- Projected Dynamical Systems (PDS) [Dupuis and Nagurney 1993, Zhang and Nagurney 1995, Cojocaru and Jonker 2004]:
  - a significant line of **independent research**
  - attracted economists, physicists, mathematicians, etc.
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Proposition (GPAW Compensated System is a PDS)

The GPAW compensated system Σ_g is a projected dynamical system governed by $\dot{z} = f_g(z) = \pi(z, f_n(z))$.

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Proposition (Existence and Uniqueness of Solutions)

The GPAW compensated system Σ_g has a unique solution for all initial conditions $z_0 \in \mathbb{R}^2$ for all $t \geq t_0$.

Region of Attraction

- Anti-windup schemes aim to enhance performance only under control saturation
- Require anti-windup schemes to enhance performance **without reducing region of attraction (ROA)**
- Let $\phi_n(t, z)$ be solution of uncompensated system Σ_n,
  $\phi_g(t, z)$ be solution of GPAW compensated system Σ_g

ROA for systems Σ_n and Σ_g:

$$R_n := \{z \in \mathbb{R}^2 \colon \phi_n(t, z) \to z_{eq0} \text{ as } t \to \infty\}$$
$$R_g := \{z \in \mathbb{R}^2 \colon \phi_g(t, z) \to z_{eq0} \text{ as } t \to \infty\}$$

Valid Anti-windup Scheme

Require $R_n \subset R_g$ to be valid anti-windup scheme.

Region of Attraction Containment

Unsaturated region: $K = \{(x,u) \in \mathbb{R}^2 \mid u_{min} < u < u_{max}\}$
Proved in ACC paper:

Proposition (ROA Containment in Unsaturated Region)

The part of the ROA of the origin of system Σ_n contained within the unsaturated region, is itself contained within the ROA of the origin of system Σ_g, ie. $(R_n \cap \bar{K}) \subset R_g$.

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Main result proved in technical report:

J. Teo and J. P. How, "Gradient projection anti-windup scheme on constrained planar LTI systems," MIT, Cambridge, MA, Tech. Rep. ACL10-01, Mar. 2010, Aerosp. Controls Lab. [Online]. Available: http://hdl.handle.net/1721.1/52600

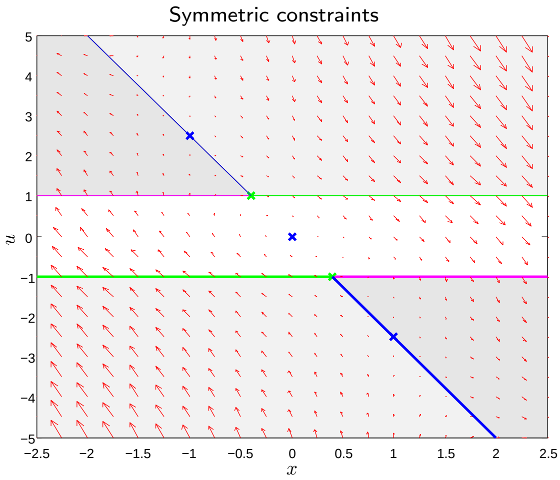
Proposition (ROA Containment in $\mathbb{R}^2$)

The ROA of the origin of system Σ_n is contained within the ROA of the origin of system Σ_g, ie. $R_n \subset R_g$.

- for **all** parameters $(a, b, c, d, u_{min}, u_{max})$ s.t. unconstrained system stable

Numerical Results

Unstable open loop plant



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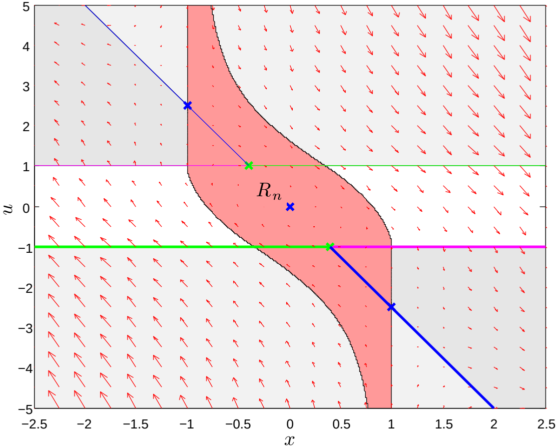
Symmetric constraints



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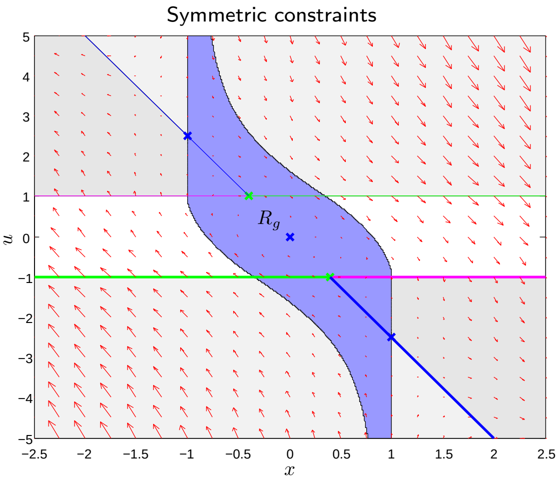

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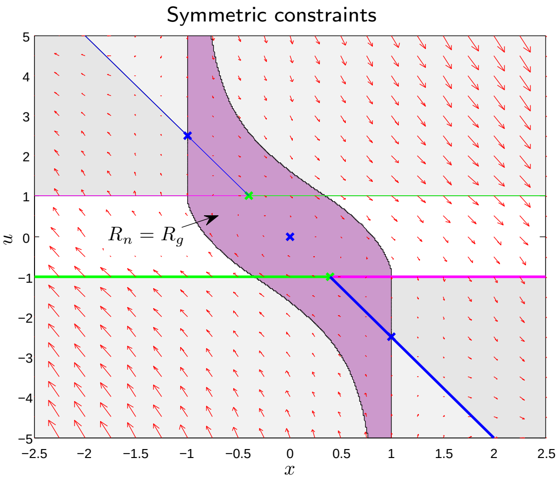

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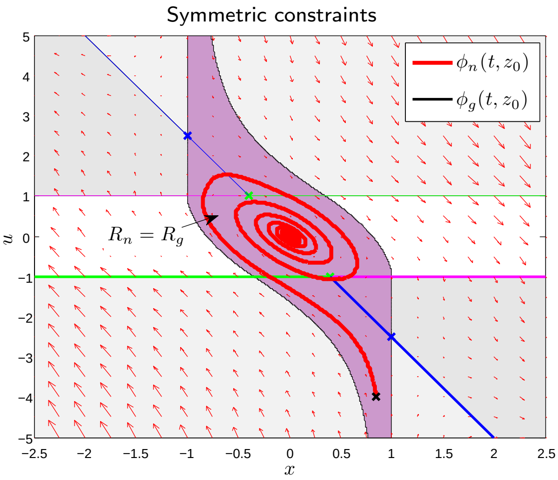

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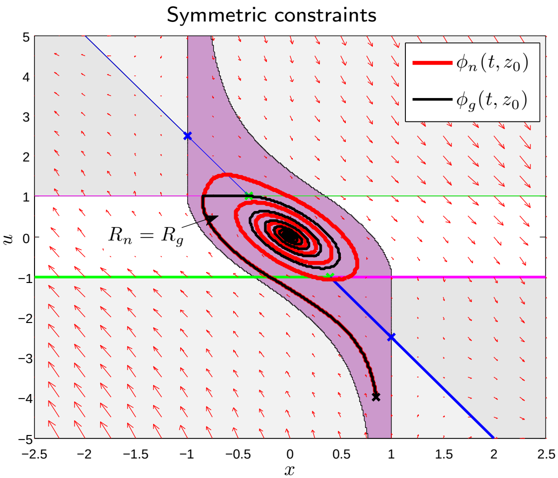

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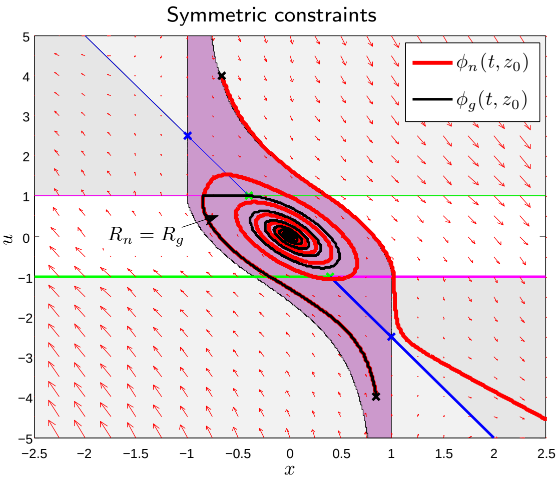

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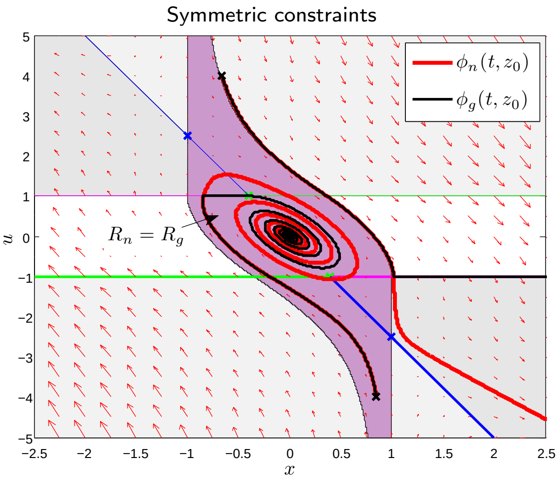

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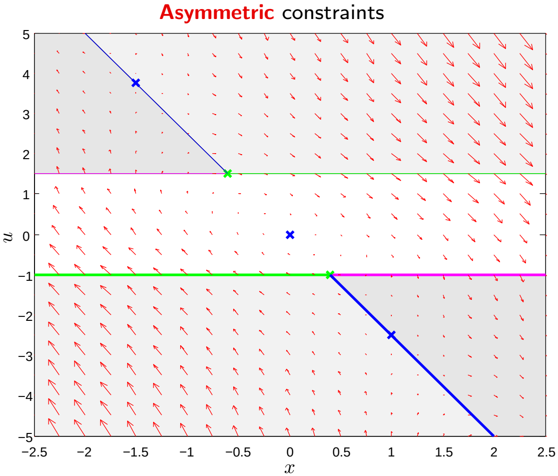

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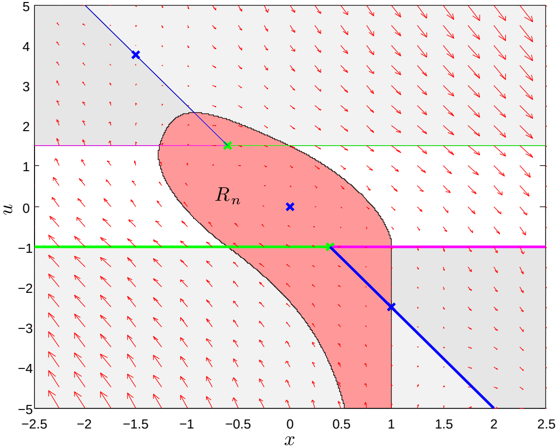
Asymmetric constraints



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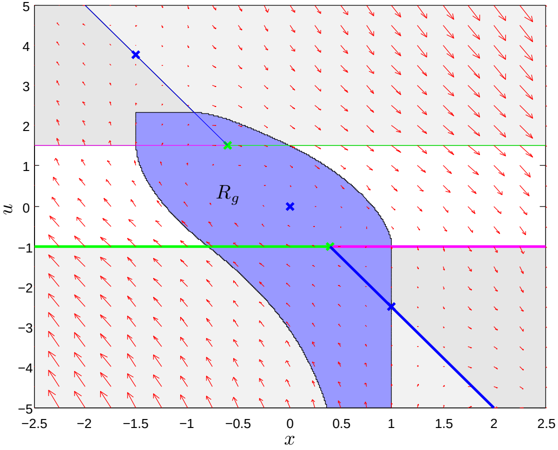

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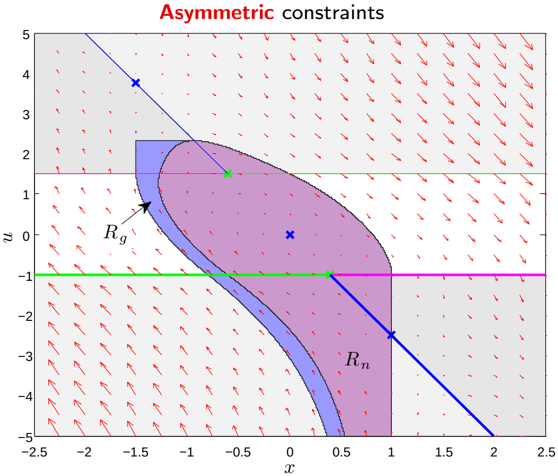

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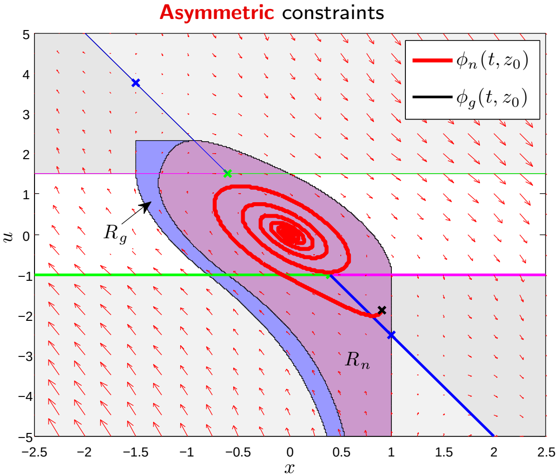

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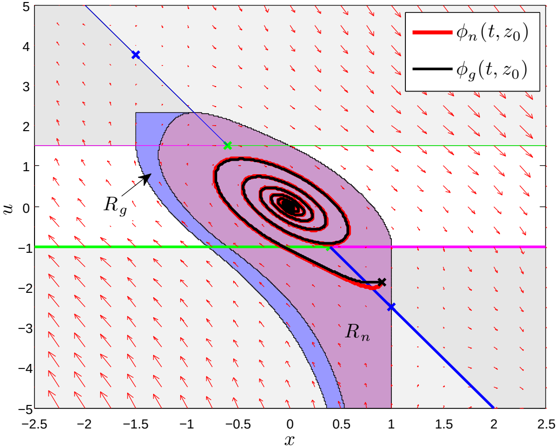

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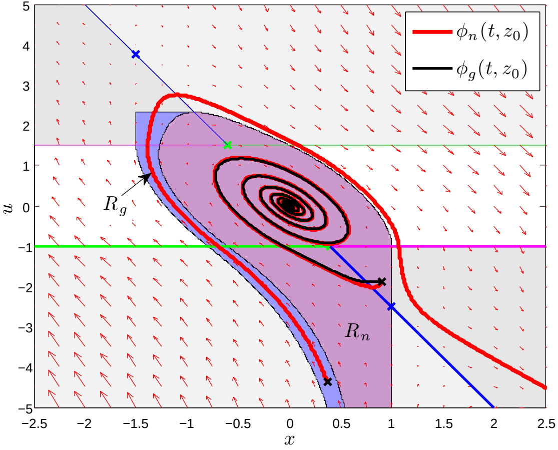

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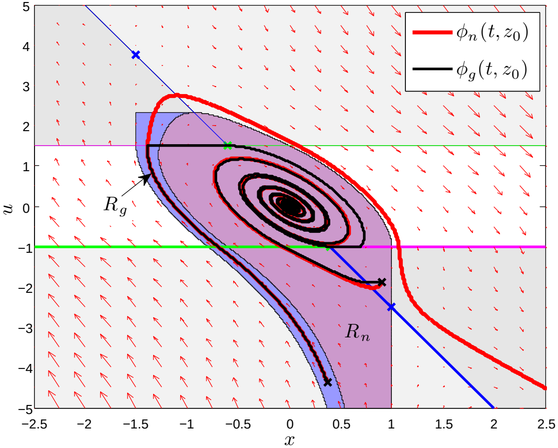

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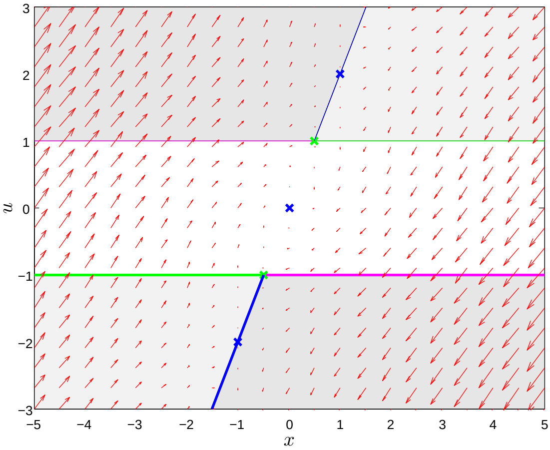
Stable plant, unstable controller



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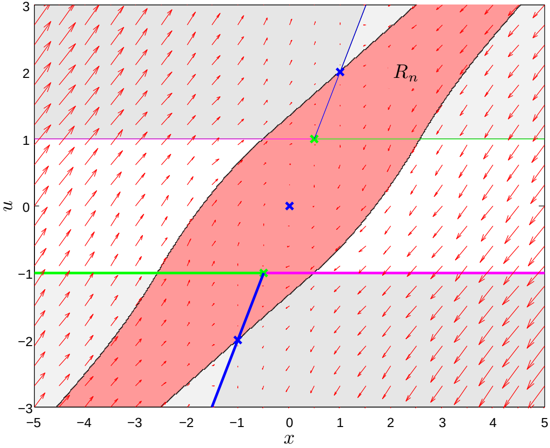

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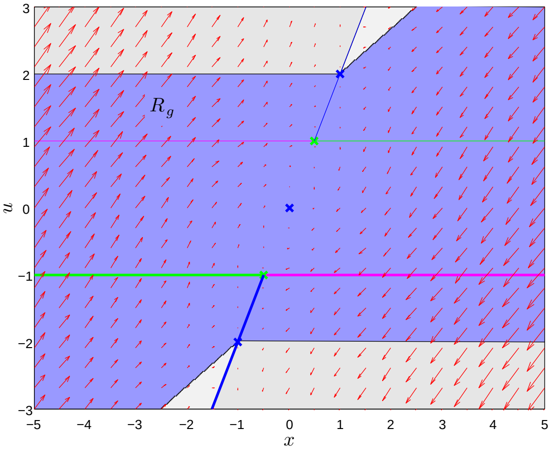

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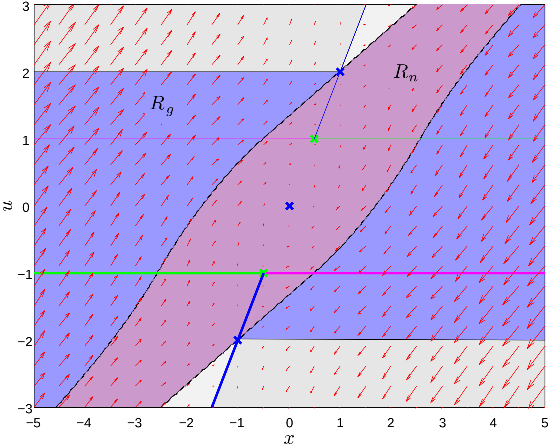

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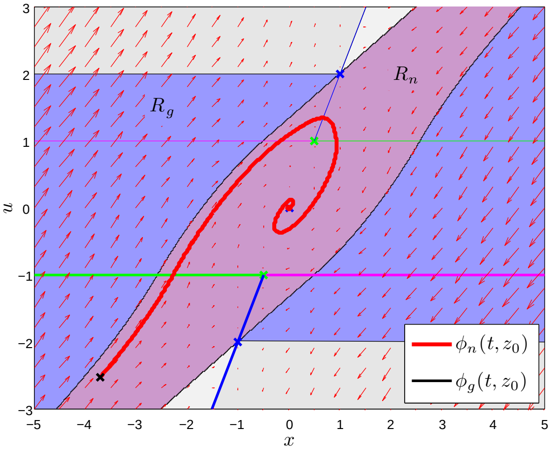

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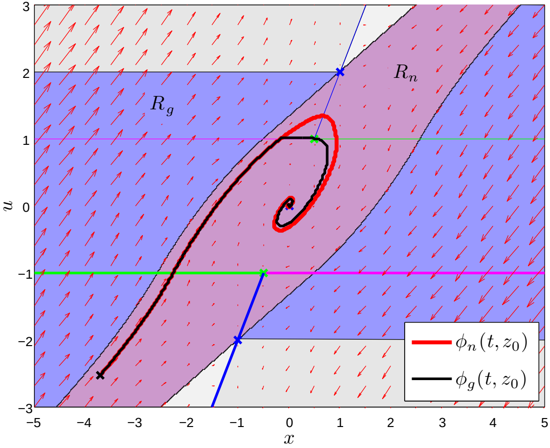

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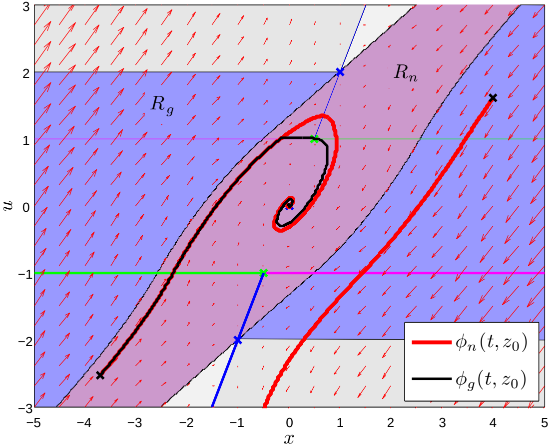

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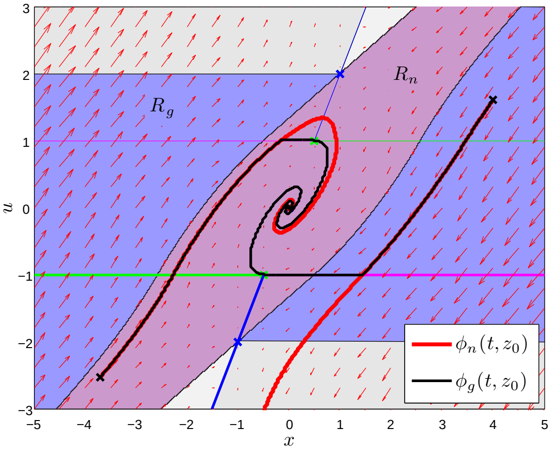

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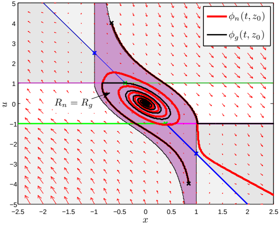

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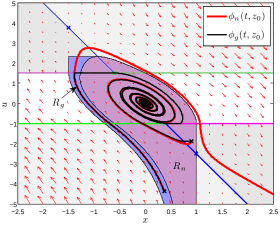

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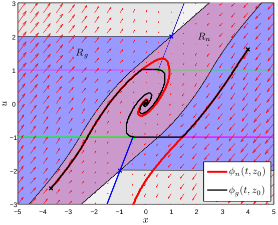

Need to Consider Asymmetric Constraints

Conjecture (Relaxing Constraints Imply ROA Enlargement)

Let R_{g1} *be ROA for some saturation limits* u_{min1}, u_{max1}*, and* R_{g2} *be ROA for* u_{min2}, u_{max2}*. If* $[u_{min1}, u_{max1}] \subset [u_{min2}, u_{max2}]$*, then* $R_{g1} \subset R_{g2}$*.*

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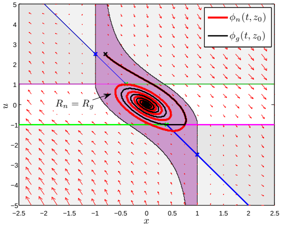
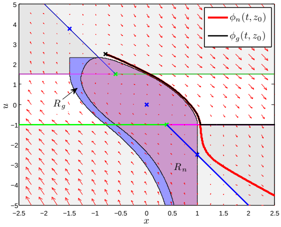
Conjecture intuitively appealing, but **WRONG!** See also [Hu et al. 2002]

Symmetric



$u_{max} = -u_{min} = 1$

Asymmetric



$u_{max} = 1.5$, $u_{min} = -1$

Not pathological

Need to consider asymmetric saturation contraints

Relative Results More Meaningful

Corollary (GAS and LES of GPAW Compensated System)

Assume **unconstrained** *nominal system is stable, and both open loop plant and nominal controller are marginally or strictly stable. Then the origin of GPAW compensated system* Σ_g *is* **globally asymptotically stable** *(GAS) and locally exponentially stable (LES).*

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Claim (GAS and LES of Nominal System)

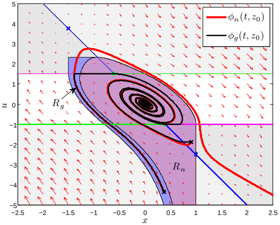
Under same hypotheses, origin of **saturated** *nominal system* Σ_n *is GAS and LES.*

- Some anti-windup results are of the form of preceding Corollary
- Such results tells **nothing** about advantages of anti-windup method
- Same result obtained as Corollary of ROA containment
- ROA containment result shows true advantage
- Propose to search for **relative** results

Solution Bounce Property

Definition (Bounce of Solution)

A "bounce" of a system's solution is the event where the solution reaches the saturation constraint boundary (∂K_+ or ∂K_-) from the unsaturated region K.



Claim (Upper Limit on Bounces of GPAW Compensated System)

For all $z_0 \in R_g$, the solution $\phi_g(t, z_0)$ of the GPAW compensated system can have at most **one** *bounce each on ∂K_+ and ∂K_-.*

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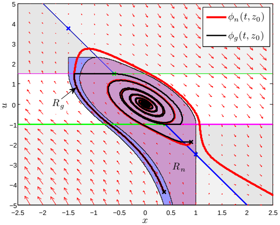

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Suggests possible "optimality in solution bounces" property that needs further study

Conclusions

- Considered simplest possible feedback system
- GPAW compensated constrained planar LTI system is a **projected dynamical system**, a significant line of independent research
- Existence and Uniqueness of solutions assured despite discontinuous vector field
- Main Result: GPAW scheme can only **maintain/enlarge** ROA
- Showed that consideration of **asymmetric** constraints necessary
- Showed **relative** results may be more meaningful
- Presented the “bounce” property

Detailed technical report (less “bounce” property):

J. Teo and J. P. How, “Gradient projection anti-windup scheme on constrained planar LTI systems,” MIT, Cambridge, MA, Tech. Rep. ACL10-01, Mar. 2010, Aerosp. Controls Lab. [Online]. Available: http://hdl.handle.net/1721.1/52600

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Questions?

Equilibrium Points

Not surprisingly,

Claim (Equilibrium point at origin z_{eq0})

The origin $z_{eq0} := (0, 0)$ is the **only** *equilibrium point of systems Σ_n and Σ_g in unsaturated region K, and it must be either a stable node or stable focus.*

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Multiple equilibria when either plant or controller is unstable:

Claim (Additional equilibria for Σ_n)

Apart from the origin z_{eq0}, the **nominal** *system Σ_n admits two additional* **isolated** *equilibrium points only when either plant or controller is* **unstable**, *and they must be saddle points.*

Claim (Additional equilibria for Σ_g)

Apart from the origin z_{eq0}, the **GPAW compensated** *system Σ_g admits a* **continuum** *of equilibria only when either plant or controller is* **unstable**.

References I

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