Hybrid Model Reference Adaptive Control for Unmatched Uncertainties

John F. Quindlen[1] Girish Chowdhary[2] Jonathan P. How[3]

Abstract— **This paper presents a hybrid model reference adaptive control approach for systems with both matched and unmatched uncertainties. This approach extends concurrent learning adaptive control to a wider class of systems with unmatched uncertainties that lie outside the space spanned by the control input, and therefore cannot be directly suppressed with inputs. The hybrid controller breaks the problem into two parts. First, a concurrent learning identification law guarantees the estimates of the unmatched parameterization converges to the actual values in a determinable rate. While this begins, a robust reference model and controller maintain stability of the tracking and matched parameterization error. Once the unmatched estimates have converged, the system exploits this information to switch to a more aggressive controller to guarantee global asymptotic convergence of all tracking, matched, and unmatched errors to zero. Simulations of simple aircraft dynamics demonstrate this stability and convergence.**

I. INTRODUCTION

Adaptive control achieves a desired level of performance for systems with uncertain dynamics by combining parameter estimation with control. Model reference adaptive control (MRAC) is a direct approach that forces the uncertain system to track a reference model with desirable performance, stabilizing the system even in the presence of destabilizing nonlinear uncertainties [1]–[3]. These baseline MRAC procedures have proven success on many different applications, but have mostly been restricted to uncertainties that fall entirely within the space spanned by the control input matrix. This assumption, known as *matched uncertainty*, means the control input can completely cancel out the uncertainty in the system. In many physical systems, this assumption may not hold and portions of the uncertainty, known as the *unmatched uncertainty*, will lie outside the span of the control input.

While the vast majority of the previous work has focused on matched uncertainty, there have been techniques specifically focused on uncertainty not in the span of the control input. Linear Matrix Inequality (LMI) techniques create robust reference models to ensure stability in the presence of unmatched uncertainty [4]. Additionally, the bi-objective adaptive [5] and sliding mode [6], [7] controllers have also been examined to improve stability with unmatched uncertainty. The L_1 approach [8] was coupled with estimates of the unmatched parameterization in a closed-loop reference model. While these methods [4]–[8] have been shown to ensure stability in the presence of both matched and unmatched uncertainty, they fail to guarantee the correct parameterization of the unmatched uncertainty will be learned.

Concurrent learning methods offer the potential to learn this uncertainty without requiring restrictive conditions. While so far limited to problems with matched uncertainty, concurrent learning model reference adaptive control (CL-MRAC) has been used to guarantee matched weight error convergence to zero while simultaneously achieving tracking error convergence to zero. Standard MRAC procedures are able to guarantee tracking error convergence, but the estimated weights will only converge to the actual values in the presence of Persistently Exciting (PE) signals [9], a restrictive condition that may not be possible in many applications. Concurrent learning methods remove this restriction and guarantee weight error convergence to zero without requiring PE signals [9], [10]. In particular, the CL-MRAC approach appends recorded state data to the baseline MRAC adaptive law. This recorded state data acts as memory for the adaptive law and enables adaptation of the weight estimates even while the instantaneous tracking error is zero. This CL-MRAC approach has been successfully demonstrated in flight tests and experiments [11], [12].

In this paper, concurrent learning methods are applied to adaptive systems in the presence of both types of uncertainty. A two stage approach is taken. A concurrent learning identification law learns the parameterization of the unmatched uncertainty. Simultaneously, a robust adaptive controller maintains stability while the unmatched estimates are learned. Once the unmatched error has fallen below a determinable, sufficiently small bound, the unmatched estimates are used to recompute the reference model. This removes the unmatched uncertainty from the reference model and a CL-MRAC control law will then guarantee tracking and matched weight error convergence to zero alongside the unmatched error. Simulations of aircraft short period dynamics demonstrate this guaranteed convergence of all three errors to zero even with non-PE reference signals.

II. SYSTEM DESCRIPTION

Consider the following uncertain dynamical system

$$\dot{x}(t) = Ax(t) + B(u(t) + \Delta(x)) + B_u\Delta_u(x) \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $u(t) \in \mathbb{R}^m$ is the corresponding control input vector. Assume $x(t)$ is available for measurement. The pair of matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are both known and controllable. This pair defines the nominal system dynamics; however, uncertainties $\Delta(x) \in \mathbb{R}^m$ and $\Delta_u(x) \in \mathbb{R}^{m_2}$ are also present in the actual

[1]Graduate Student, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology

[2]Assistant Professor, Department of Mechanical and Aerospace Engineering, Oklahoma State University

[3]Richard C. Maclaurin Professor of Aeronautics and Astronautics, Aerospace Controls Laboratory (Director), MIT

dynamics. Here $\Delta(x)$ encompasses all the matched uncertainties that lie within the span of B while the unmatched uncertainties $\Delta_u(x)$ lie outside the span of B.

Remark 1 The orthogonal complement of B, $B_u \in \mathbb{R}^{n\times m_2}$, can be calculated directly from the left nullspace of B. Here, B_u fully spans this left nullspace so that $B_u^T B = 0$.

The matched and unmatched uncertainties are both modeled as structured uncertainty in this paper. Neural networks or radial basis functions can be used to deal with unstructured forms of the matched uncertainty [9], but are not considered.

The structured matched uncertainty is parameterized by an unknown weight matrix $W^* \in \mathbb{R}^{k\times m}$ and a known regressor vector $\phi(x) \in \mathbb{R}^k$

$$\Delta(x) = W^{*T}\phi(x) \tag{2}$$

where $\Delta(x)$ can be either a linear or nonlinear basis vector without loss of generality. Let $\phi(x)$ be Lipschitz continuous to ensure the existence and uniqueness of the solution [1].

Assumption 1 The unmatched uncertainty is parameterized by an unknown matrix $\Omega^* \in \mathbb{R}^{n\times m_2}$ and the known basis vector is simply the state vector x.

$$\Delta_u(x) = \Omega^{*T}x \tag{3}$$

Additionally, the unknown weighting matrix Ω^* has a known upper bound, $||\Omega^*|| \leq \Omega^*_{max}$.

Remark 2 While Assumption 1 requires a linear basis vector, this is done in order to relax the restrictive assumption used in previous works [4], [5], [8] that requires the unmatched uncertainty to be bounded from above.

In the next sections, estimates of the unmatched uncertainties are used to update the reference model. A linear reference model is used in this paper, a common assumption used in the vast majority of adaptive control works. This requires that the basis of $\Delta_u(x)$ be the state vector.

III. ADAPTIVE IDENTIFICATION

By definition, unmatched uncertainty lies outside the span of the control matrix B; therefore, it cannot be directly canceled out with control input $u(t)$. While this does prevent a direct adaptive control approach from canceling the unmatched uncertainty by itself, the unmatched terms can still be learned online. In this section, the proposed concurrent learning-based adaptive identification law is shown to guarantee the global exponential convergence of the estimation error of the unmatched terms to zero.

A. Concurrent Learning Identification Law

The contribution of the unmatched uncertainty terms can be measured in the system response. Rearranging (1), the unmatched uncertainty $\Delta_u(x)$ can be rewritten as

$$\Delta_u(x) = B_u^{\dagger}(\dot{x} - Ax - Bu) \tag{4}$$

where $B_u^{\dagger}$ is the right pseudo-inverse of B_u. The orthogonality of B and B_u discussed in Remark 1 means the matched uncertainty will not corrupt the measurement of the unmatched terms.

Remark 3 In order to measure the unmatched uncertainty $\Delta_u(x)$ in (4), either the actual state derivative $\dot{x}(t)$ or an estimate of that derivative must be available. If $\dot{x}(t)$ itself is not immediately available, a fixed point smoother can be used to reliably and accurately estimate the state derivative even in the presence of noise. This smoothing method has been successfully proven and experimentally validated to yield good estimates of $\dot{x}(t)$ [11], [12]. The subsequent sections of this paper will assume the actual derivative $\dot{x}(t)$ is measurable; however, these results can be directly applied to the aforementioned cases without loss of generality.

The unknown weighting matrix Ω^* is estimated using a concurrent learning identification law. This gradient descent algorithm uses instantaneous measurements of the unmatched uncertainty from (4) concurrently with a fixed, finite number $\overline{p}$ of previous measurements. The resulting update equation is

$$\dot{\hat{\Omega}} = -\Gamma_u x(\hat{\Omega}^T x - \Delta_u(x))^T - \Gamma_{u_c}\sum_{j=1}^{\overline{p}} x_j\sigma_j^T \tag{5}$$

where $\sigma_j = \hat{\Omega}^T x_j - \Delta_u(x_j)$. The learning rates Γ_u and Γ_{u_c} are positive constants. It is also assumed that $\hat{\Omega}(0) = 0$.

The key benefit of this approach is that it guarantees parameter convergence without requiring PE.This is achieved through the time history of saved measurements, $\sum_{j=1}^{\overline{p}} x_j\sigma_j^T$. In order to guarantee convergence, the following condition must be met by the history stack.

Condition 1 The stack of state vectors $Z_x(t) = \left[x_1, ..., x_{\overline{p}}\right]$ must be linearly independent, that is $\texttt{rank}(\texttt{Z}_\texttt{x}) = \texttt{dim}(\texttt{x}) = \texttt{n}$ and $\overline{p} \geq n$.

Remark 4 Condition 1 ensures matrix $M_x = (Z_x(t)Z_x^T(t)) > 0$ for all time t; therefore, the minimum singular value of the matrix is positive, $\lambda_{min}(M_x) > 0$.

Remark 5 The time history stack should be updated according to the singular value maximizing (SVM) algorithm [13] in order to ensure Condition 1 is met.

Further information and proofs for this algorithm can be found in [9], [13].

B. Lyapunov Analysis

The concurrent learning identification law guarantees the convergence of the unmatched weight estimates $\hat{\Omega}$ to the actual values Ω^* without requiring PE signals. The following subsection proves this statement and provides a bound on the convergence rate.

The unmatched weight estimate error is $\tilde{\Omega} = \hat{\Omega} - \Omega^*$.

Theorem 1 Consider the system (1) and (3). If the unmatched weight estimates are updated according to the identification law (5) and the SVM algorithm [13], then the estimation error will globally exponentially converge to the zero solution $\tilde{\Omega} = 0$.

Proof: Let $V_{\tilde{\Omega}}$ be a differentiable, positive definite, radially unbounded Lyapunov function candidate that describes the convergence of the unmatched weight estimation error.

$$V_{\tilde{\Omega}} = \frac{1}{2}\mathtt{trace}(\tilde{\Omega}^T \Gamma_u^{-1} \tilde{\Omega}) \tag{6}$$

Since $V_{\tilde{\Omega}}$ is quadratic, there exists known α and β such that

$$\alpha||\tilde{\Omega}||^2 \leq V_{\tilde{\Omega}} \leq \beta||\tilde{\Omega}||^2 \tag{7}$$

where $\alpha = \frac{1}{2}\lambda_{min}(\Gamma_u^{-1})$ and $\beta = \frac{1}{2}\lambda_{max}(\Gamma_u^{-1})$. Differentiating with respect to time gives

$$\begin{aligned} \dot{V}_{\tilde{\Omega}} &= \mathtt{trace}(\tilde{\Omega}^T \Gamma_u^{-1} \dot{\tilde{\Omega}}) \\ &\leq -\mathtt{trace}(\tilde{\Omega}^T [\Gamma_u^{-1} \Gamma_{u_c} M_x] \tilde{\Omega}) \end{aligned} \tag{8}$$

Note that matrix $\Gamma_u^{-1}\Gamma_{u_c}M_x > 0$ for all time t since the SVM method ensures $M_x > 0$ and Γ_u and Γ_{u_c} are positive constants. Therefore, the derivative can be more compactly written as the following negative definite expression

$$\dot{V}_{\tilde{\Omega}} \leq -\gamma||\tilde{\Omega}||^2 \tag{9}$$

where $\gamma > 0$. Thus, the global exponential convergence to the solution $\tilde{\Omega} = 0$ is proven. ■

The global exponential convergence of the $\tilde{\Omega}$ dynamics guarantees that the actual Ω^* will be identified even without a PE signal. Not only is the convergence guaranteed, but the rate of convergence can also be determined.

Corollary 2 The estimation error $\tilde{\Omega}(t)$ at any time t can be explicitly bounded from above by a known function of time. Therefore, a conservative approximation of the unmatched weight error can be calculated online.

Proof: From the results in Theorem 1, the unmatched weight error can be bounded by an exponential function with positive constants k_1 and k_2.

$$||\tilde{\Omega}(t)|| \leq k_1||\tilde{\Omega}(0)||e^{-k_2 t} \tag{10}$$

Here,

$$k_1 = \sqrt{\frac{\beta}{\alpha}} = \sqrt{\frac{\lambda_{max}(\Gamma_u^{-1})}{\lambda_{min}(\Gamma_u^{-1})}} = 1 \tag{11}$$

$$k_2 = \frac{\lambda_{min}(\Gamma_u^{-1}\Gamma_{u_c}M_x)}{2\alpha} = \Gamma_{u_c}\lambda_{min}(M_x) \tag{12}$$

Since $\hat{\Omega}(0) = 0$ and Ω^* is upper bounded by known Ω^*_{max}, the initial estimation error is also upper bounded, $||\tilde{\Omega}(0)|| \leq \Omega^*_{max}$. The unmatched error $\tilde{\Omega}$ at any time t can then be upper bounded by the known exponential function.

$$||\tilde{\Omega}(t)|| \leq \Omega^*_{max} e^{-\Gamma_{u_c}\lambda_{min}(M_x)t} \tag{13}$$

■

IV. HYBRID MODEL REFERENCE ADAPTIVE CONTROL

This section exploits the guaranteed convergence and upper bound on the convergence rate from the previous result to formulate a hybrid direct-indirect adaptive control approach. The adaptive controller suppresses the matched uncertainties and tracks a stable reference model in a direct manner; however, this reference model is updated with an estimate of the unmatched uncertainty. The following results will then prove the guaranteed convergence without PE of not only the unmatched estimation error, but also the matched estimation error and the reference model tracking error.

A. Reference Model Selection

The goal of the adaptive controller is to force the system to track a stable reference system with favorable performance characteristics. In this paper, this reference model is assumed to be linear and therefore the desired system performance is defined by a set of eigenvalues. Let $x_m(t) \in \mathbb{R}^n$ be the reference state the adaptive system tracks. The reference model is given by

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t) \tag{14}$$

where the external reference command $r(t) \in \mathbb{R}^{m_r}$, $m_r \leq m$, is assumed to be piecewise continuous and bounded. The control system then attempts to minimize the state tracking error between the reference model and the actual system, $e(t) = x_m(t) - x(t)$, with control inputs $u(t) \in \mathbb{R}^m$.

In the absence of uncertainty, a feedback controller K can be chosen to place the closed loop poles at a desired location so that $A_m = A - BK$. Even for the traditional adaptive control problem, the fact that the uncertainty falls completely within the span of the control matrix allows the control inputs to track this reference model. However, in the presence of unmatched uncertainty, the control inputs themselves can no longer perfectly track the reference model and this breaks down.

Instead, the unmatched weight estimates $\hat{\Omega}(t)$ and the corresponding upper bound on $\tilde{\Omega}(t)$ are used to update the reference model. At every time t, not only is an estimate $\hat{\Omega}(t)$ available, but Corollary 2 states that a conservative estimate of $||\tilde{\Omega}(t)||$ can also be calculated.

Two different reference models are used to guarantee stability and convergence of the system. Algorithm 1 details the switching approach. In effect, the first reference model is used to maintain stability and robustness as the unmatched terms are estimated using the identification law in (5). This controller and reference model must be selected to meet Condition 2 to ensure stability in the presence of unmatched uncertainty. Once the estimate has converged to a solution at time t_k, the system switches to a new reference model that satisfies Condition 3 by incorporating $\hat{\Omega}(t_k) \approx \Omega^*$.

The trigger for the reference model switch is given on line 4 of Algorithm 1. Once $||\tilde{\Omega}(t)||$ has passed below a sufficiently small bound ϵ at time t_k, the unmatched error is $\tilde{\Omega}(t_k) \approx 0$ and estimate $\hat{\Omega}(t_k) \approx \Omega^*$. Then the system can switch from A_{m_1}, K_1 to tighter performance with A_{m_2},

K_2. Additionally, this switch does not affect stability of the system as both A_{m_1} and A_{m_2} are robust to conservative estimates of the unmatched uncertainty.

Condition 2 K_1 and A_{m_1} are chosen such that $A_{m_1} = A - BK_1$ is Hurwitz, $A_{m_1}^T P_1 + P_1 A_{m_1} = -Q$ with $Q = Q^T > 0$, and the system is robust to unmatched uncertainty, that is $-\lambda_{min}(Q)||e(t)|| > 2||P_1 B_u||\Omega^*_{max}||x(t)||$. Note that $||\Omega^*|| \leq \Omega^*_{max}$ is known from Assumption 1.

Remark 6 Since the upper bound $||\Omega^*|| \leq \Omega^*_{max}$ is known from Assumption 1, a suitable reference model can be selected using sector bounds and Linear Matrix Inequalities [4], [5] from robust control theory to ensure Condition 2 holds.

Condition 3 K_2 and A_{m_2} are chosen such that $A_{m_2} = A + B_u \hat{\Omega}^T(t_k) - BK_2$ is Hurwitz, $A_{m_2}^T P_2 + P_2 A_{m_2} = -Q$ with $Q = Q^T > 0$, and the system is robust to any remaining unmatched error, $-\lambda_{min}(Q)||e(t)|| > ||P_2 B_u||\ ||\tilde{\Omega}(t_k)||\ ||x(t)||$. Note that any remaining unmatched estimate error $||\tilde{\Omega}(t_k)||$ can be conservatively bounded from Corollary 2, but threshold ϵ should be chosen such that $||\tilde{\Omega}(t_k)|| \approx 0$.

Algorithm 1 Reference Model Switch

1: Initial unmatched estimation error given by $||\tilde{\Omega}(0)|| = \Omega^*_{max}$
2: Select A_{m_1} and K_1 according to Condition 2
3: while time $t < \infty$
4: **if** $||\tilde{\Omega}(t_k)|| < \epsilon$ **then**
5: Select A_{m_2} and K_2 according to Condition 3
6: **end if**

B. Concurrent Learning MRAC

The goal of the concurrent learning adaptive controller is to track the desired reference model in the presence of uncertainty using control inputs. The control input is a combination of three parts:

$$u = u_{rm} + u_{pd} - u_{ad} \tag{15}$$

a feedforward term $u_{rm} = K_r r(t)$ where $B_m = BK_r$, a feedback term $u_{pd} = -Kx$, and the adaptive control input u_{ad}. Here K is the appropriate K_1 or K_2 taken from Algorithm 1.

In the preceding sections, the focus has been on dealing with unmatched uncertainties. Now, the matched uncertainties are learned and suppressed using the adaptive control inputs. Since the weighting matrix W^* is unknown, an estimate $\hat{W} \in \mathbb{R}^{k\times m}$ is used instead with error $\tilde{W} = \hat{W} - W^*$. This matrix is multiplied by the known regressor vector $\phi(x)$ to give the control input:

$$u_{ad} = \hat{W}^T \phi(x) \tag{16}$$

While both the matched and unmatched terms are linear in parameters, the basis vector $\phi(x)$ for the matched uncertainty does not have to be linear. Additionally, the basis vector is assumed to have a known structure. While it will not be addressed in this paper, the concurrent learning algorithm still holds for systems with unstructured uncertainty [9].

The major advantage of a CL-MRAC controller is that the adaptive law enforces stability and simultaneously guarantees tracking error e and matched error $\tilde{W}$ convergence to zero without requiring persistency of excitation. Just like the identification law in (5), the CL adaptive law incorporates a time history stack of stored data points. Instead of unmatched $\Delta_u(x)$, now $\Delta(x)$ is measured using the system response:

$$\Delta(x) = B^\dagger(\dot{x} - Ax - Bu) \tag{17}$$

where $B^\dagger$ is the right pseudo-inverse $B^\dagger = (B^T B)^{-1} B^T$. Mentioned in Remark 3, this paper will work under the assumption that $\dot{x}(t)$ is measurable and therefore $\Delta(x)$ is directly obtainable from (17). As shown in previous proofs, flight tests, and demonstrations [9], [11], [12], $\dot{x}(t)$ can be obtained through a fixed point smoother and the same results will apply without loss of generality.

The concurrent learning adaptive law is given as

$$\dot{\hat{W}} = -\Gamma\phi(x)e^T PB - \Gamma_c \sum_{i=1}^{\overline{p}_2} \phi(x_i)\epsilon_i^T \tag{18}$$

where learning rates Γ and Γ_c are positive scalars and $\epsilon_i = u_{ad,i} - \Delta_i = \tilde{W}^T\phi(x_i)$. The symmetric positive definite matrix P is taken from the appropriate P_1, P_2 from Algorithm 1. A fixed, finite number $\overline{p}_2$ of measurements is used alongside the instantaneous tracking error. Just as in Section III, the time history stack guarantees parameter convergence without requiring PE signals. In order to do so, a similar condition to Condition 1 must hold:

Condition 4 The stack of state vectors $Z_\phi(t) = \left[\phi(x_1), ..., \phi(x_{\overline{p}})\right]$ must be linearly independent, that is $\mathtt{rank}(\mathtt{Z}_\phi) = \mathtt{dim}(\phi(\mathtt{x})) = \mathtt{k}$ and $\overline{p}_2 \geq k$.

Remark 7 Condition 4 ensures matrix $M_\phi = (Z_\phi(t) Z_\phi^T(t) > 0$ for all time t; therefore, the minimum singular value of the matrix is positive, $\lambda_{min}(M_\phi) > 0$. The time history stack should be updated according to the singular value maximizing (SVM) algorithm [13] to ensure Condition 4 is met.

Just as mentioned in the identification section, more information and a proof of this algorithm can be found in [9], [13]. Most importantly, this method ensures Condition 4 is met and increases the minimum singular value of M_ϕ.

C. Lyapunov Analysis

While the unmatched uncertainty adversely affects the tracking error e and therefore the matched estimate $\hat{W}$, the system will remain stable. The hybrid CL-MRAC approach in Algorithm 1 and Equation 18 actually guarantees the state tracking and matched weight estimate errors will asymptotically converge to $e, \tilde{W} = 0$ in a finite time without requiring PE inputs. This result is shown in two parts corresponding to the two steps from Algorithm 1.

First, consider the initial reference model A_{m_1} and the robust controller K_1 before the switch at time t_k.

Theorem 3 Consider the system defined by (1-3). If the reference model (14) is set to A_{m_1} in Algorithm 1 and the matched weights are updated according to (18), then the state tracking error e and matched weight error $\tilde{W}$ are globally stable.

Proof: Let V_1 be a differentiable, positive definite, radially unbounded Lyapunov function candidate that describes the reference model tracking error and matched weight estimation error:

$$V_1 = \frac{1}{2}e^T P_1 e + \frac{1}{2}\mathtt{trace}(\tilde{W}^T \Gamma^{-1} \tilde{W}). \tag{19}$$

The error dynamics are given by

$$\begin{aligned} \dot{e} &= \dot{x}_m - \dot{x} \\ &= A_{m_1} e + B\tilde{W}^T \phi(x) - B_u \Omega^{*T} x \end{aligned} \tag{20}$$

Differentiating V_1 with respect to time gives

$$\begin{aligned} \dot{V}_1 &= \frac{1}{2}\dot{e}^T P_1 e + \frac{1}{2}e^T P_1 \dot{e} + \mathtt{trace}(\tilde{W}^T \Gamma^{-1} \dot{\tilde{W}}) \\ &= -\frac{1}{2}e^T Q e - e^T P_1 B_u \Omega^{*T} x \\ &\quad - \Gamma^{-1}\Gamma_c \mathtt{trace}(\tilde{W}^T \sum_{i=1}^{\overline{p}_2} \phi(x_i)\phi(x_i)\tilde{W}) \end{aligned} \tag{21}$$

Since Algorithm 1 enforces Condition 2 then

$$\frac{1}{2}\lambda_{min}(Q)||e|| > ||P_1 B_u \Omega^{*T} x|| \tag{22}$$

Therefore, $\dot{V}_1 \leq 0$ for all e and $\tilde{W}$ and globally stable since V_1 is radially unbounded. ∎

Now consider the system after the unmatched estimation error has converged below an arbitrarily small ϵ $||\tilde{\Omega}(t)|| < \epsilon$. This triggers the switch from A_{m_1} to A_{m_2} in Algorithm 1. This switch occurs at time t_k and $\hat{\Omega}_k = \hat{\Omega}(t_k) \approx \Omega^*$.

Theorem 4 Consider the system defined by (1-3). The reference model (14) is set to A_{m_2} in Algorithm 1. If the matched weights are updated according to (18) and the SVM algorithm [13], then the solution $e = 0, \tilde{W} = 0$ is global exponentially stable.

Proof: Let V_2 be a differentiable, positive definite, radially unbounded Lyapunov function candidate that describes the reference model tracking error and matched weight estimation error.

$$V_2 = \frac{1}{2}e^T P_2 e + \frac{1}{2}\mathtt{trace}(\tilde{W}^T \Gamma^{-1} \tilde{W}) \tag{23}$$

Unlike the previous A_{m_1}, A_{m_2} incorporates unmatched estimate $\hat{\Omega}_k$. The new form of the tracking error dynamics reflects this where $\tilde{\Omega}_k = \hat{\Omega}_k - \Omega^*$.

$$\dot{e} = A_{m_2} e + B\tilde{W}^T \phi(x) + B_u \tilde{\Omega}_k^T x \tag{24}$$

Differentiating V_2 with respect to time gives

$$\begin{aligned} \dot{V}_2 &= \frac{1}{2}\dot{e}^T P_2 e + \frac{1}{2}e^T P_2 \dot{e} + \mathtt{trace}(\tilde{W}^T \Gamma^{-1} \dot{\tilde{W}}) \\ &= -\frac{1}{2}e^T Q e + e^T P_2 B_u \tilde{\Omega}_k^T x \\ &\quad - \Gamma^{-1}\Gamma_c \mathtt{trace}(\tilde{W}^T \sum_{i=1}^{\overline{p}_2} \phi(x_i)\phi(x_i)\tilde{W}) \end{aligned} \tag{25}$$

A trigger limit ϵ was chosen to be sufficiently small such that conservative bound $||\tilde{\Omega}(t)|| < \epsilon \approx 0$. Therefore, $\hat{\Omega}(t_k) \approx \Omega^*$ and $||e^T P_2 B_u \tilde{\Omega}_k^T x|| \approx 0$. The time derivative $\dot{V}_2$ then reduces to:

$$\dot{V}_2 = -\frac{1}{2}e^T Q e - \Gamma^{-1}\Gamma_c \mathtt{trace}(\tilde{W}^T \sum_{i=1}^{\overline{p}_2} \phi(x_i)\phi(x_i)\tilde{W}) \tag{26}$$

As both terms in (26) are negative quadratics, $\dot{V}_2$ asymptotically converges to the solution $e = 0, \tilde{W} = 0$. Since V_2 was radially unbounded, this means the zero solution is globally exponentially stable. ∎

Corollary 5 From Theorems 3 and 4, the state tracking error and match weight estimation error are global asymptotically stable.

Proof: Theorem 3 states the adaptive system is globally stable for a finite time period $0 \leq t < t_k$ before switching to the second system at time t_k. Theorem 4 states this second system is global exponentially stable. Since there is only one discrete jump from global stability to stronger exponential stability, the total system can be upper bounded by a common, global asymptotically stable Lyapunov function [14]. ∎

V. SIMULATION

The hybrid concurrent learning adaptive control approach is demonstrated by a simulation of aircraft short period dynamics. The nominal model (A, B) is known, stable, and controllable. Matrix B_u is taken straight from the orthogonal complement of the known B.

$$A = \begin{bmatrix} -1.6 & 1 \\ -2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0.3 \\ 4 \end{bmatrix} \quad B_u = \begin{bmatrix} -4 \\ 0.3 \end{bmatrix} \tag{27}$$

The two state variables $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ are controlled from a single control input $u(t)$. In this problem, the matched uncertainty is parameterized by a linear basis $\phi(x) = x$ and will destabilize the system in the absence of an adaptive controller.

$$W^* = \begin{bmatrix} 1.7402 \\ 0.1367 \end{bmatrix} \qquad \Omega^* = \begin{bmatrix} 0.1305 \\ 0.0103 \end{bmatrix} \tag{28}$$

The learning rates are set to $\Gamma = 3$, $\Gamma_u = 2$, and $\Gamma_c = \Gamma_{u_c} = 0.3$. The number of data points stored in

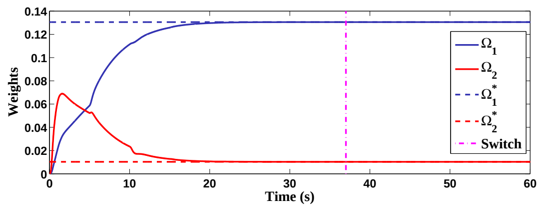

Fig. 1. Estimation of the unmatched weights from the concurrent learning identification law (5).

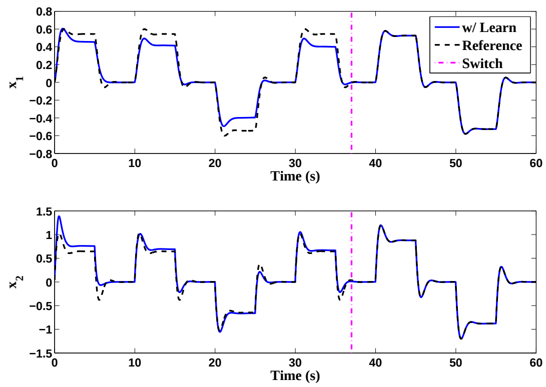

Fig. 2. State response of the actual and reference systems. Note the improved tracking error convergence after the reference model switch.

the identification law (5) and control law (18) are both capped at 20 points, $\overline{p} = \overline{p}_2 = 20$. Non-PE unit step reference commands are turned off and on every 5 seconds. Figure 1 shows the unmatched weight estimate convergence. The unmatched weight estimates converge below the chosen trigger $||\tilde{\Omega}(t)|| \leq 2 \times 10^{-6}$ in 37.2 seconds. The reference model then switches from A_{m_1} to A_{m_2}.

The state tracking and matched weight estimate responses highlight the improvement in convergence caused by the switch from A_{m_1} to A_{m_2}. The state response is shown in Figure 2. Before the switch, the actual system cannot perfectly track the reference states due to the unmatched uncertainty. Once the reference model switches at $t_k = 37.2$ seconds, the error converges to zero. Likewise, Figure 3 shows the matched weight adaptation and the distinct improvement resulting from the switch. The unmatched uncertainty corrupts the tracking error, thus causing the matched weights to fluctuate around the actual values. Once the learned model is incorporated, the tracking error converges to zero and the matched estimation error follows suit. This hybrid concurrent learning approach guarantees the errors e, $\tilde{W}$, and $\tilde{\Omega}$ will all converge to zero in a finite time without requiring a PE signal.

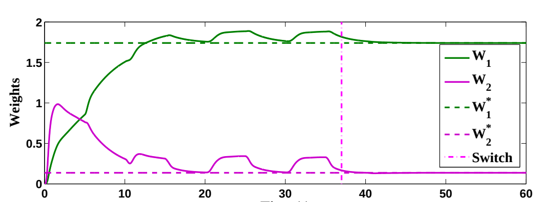

Fig. 3. Estimation of the matched weights from the concurrent learning adaptive control law (18). Note the improved weight estimate error convergence after the reference model switch.

VI. CONCLUSION

In this paper, we presented a hybrid direct-indirect adaptive control framework to exploit concurrent learning to address a class of problems with both matched and unmatched sources of uncertainty. Here we used the insight that even if the unmatched uncertainty cannot be directly suppressed with control inputs, it can still be learned online. To that effect, a concurrent learning identification law learns the unmatched parameterization with a known convergence rate while a robust adaptive control law maintains stability. After the unmatched error drops below a sufficiently small threshold, the unmatched estimate is used to update the reference model and subsequently guarantees global asymptotic convergence of all tracking, matched weight, and unmatched weight errors to zero. Numerical simulations of a simple aircraft model demonstrate the improvement afforded by this hybrid approach.

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