Supplementary Document for: Health Aware Planning Under Uncertainty for Collaborating Heterogeneous Teams of Mobile Agents

N. Kemal Ure[*], Girish Chowdhary,[†] Jonathan P. How,[‡] John Vian[§]

October 4, 2013

Proof of Theorem 2

Proof. Let $\zeta = [e, \texttt{vec}(K), K_r]^T$ and define the following Lyapunov candidate,

$$V(\zeta) = \frac{1}{2}e^T P e + \frac{1}{2}tr(\tilde{K}^T \Gamma_x^{-1} \tilde{K}) + \frac{1}{2}tr(\tilde{K_r}^T \Gamma_r^{-1} \tilde{K_r}). \tag{1}$$

It is possible to bound the Lyapunov candidate above and below with the following positive definite functions.

$$\frac{1}{2}\min(\lambda_{\min}(P), \lambda_{\min}(\Gamma_x^{-1}), \Gamma_r^{-1})\|\zeta\|^2 \leq V(\zeta) \\ \leq \frac{1}{2}\max(\lambda_{\max}(P), \lambda_{\max}(\Gamma_x^{-1}), \Gamma_r^{-1})\|\zeta\|^2$$

Let $[t_1, t_2, ..., t_p], t_{i+1} > t_i$ be the sequence of times where each data was recorded. Taking the derivative of the Lyapunov candidate along trajectories

[*]PhD. Candidate, Aerospace Controls Laboratory (ACL), Department of Aeronautics and Astronautics, Massachusetts Institute of Technology (MIT), Cambridge, MA, ure@mit.edu

[†]Assistant Professor, Oklahoma State University, girish.chowdhary@okstate.edu

[‡]Richard C. Maclaurin Professor of Aeronautics and Astronautics, ACL (Director), MIT, jhow@mit.edu

[§]Technical Fellow at Boeing Research and Technology, Seattle, john.vian@boeing.com

of the system for each interval $[t_i, t_{i+1}]$ and performing simplifications, we get:

$$\dot{V}(\zeta) = -\frac{1}{2}e^T Q e + (\tilde{K}^T x e^T + \tilde{K_r}^T r e^T)P(B - \hat{B})$$
$$-tr(\tilde{K}^T \sum_{j=1}^{p} x_j \hat{\epsilon}_{K_j}^T) - tr(\tilde{K_r}^T \sum_{j=1}^{p} r_j \hat{\epsilon}_{K_r}^T)$$

Define, $\tilde{\epsilon} = \hat{\epsilon} - \epsilon, \tilde{B} = \hat{B} - B$,

$$\dot{V}(\zeta) = -\frac{1}{2}e^T Q e - (\tilde{K}^T x e^T + \tilde{K_r}^T r e^T)P\tilde{B}$$
$$-tr(\tilde{K}^T \sum_{j=1}^{p} x_j x_j^T \tilde{K}^T) - tr(\tilde{K_r}^T \sum_{j=1}^{p} r_j r_j^T \tilde{K_r}^T)$$
$$-tr(\tilde{K}^T \sum_{j=1}^{p} x_j \tilde{\epsilon}_{K_j}^T) - tr(\tilde{K_r}^T \sum_{j=1}^{p} r_j \tilde{\epsilon}_{K_r}^T).$$

Note that the matrix $\Omega = \sum_{j=1}^{p} x_j x_j^T$ is positive definite. Bounding the inequality above using norms and the triangle inequality,

$$\begin{aligned} \dot{V}(\zeta) &\leq -\frac{1}{2}\lambda_{\min}(Q)\|e\|^2 - \frac{1}{2}\lambda_{\min}(\Omega)\|\tilde{K}\|^2 \\ &-\frac{1}{2}(\sum_{j=1}^{p} r_j^2)\|\tilde{K_r}\|^2 + \|\tilde{K}\|\|x_{rm}\|\|e\|\|P\|\|\tilde{B}\| \\ &+\|\tilde{K}\|\|e\|^2\|P\|\|\tilde{B}\| + \|\tilde{K_r}\|\|r\|\|e\|\|P\|\|\tilde{B}\| \\ &+\|\tilde{K}\|\|P\|\|\sum_{j=1}^{p} x_j \tilde{\epsilon}_K\| + \|\tilde{K_r}\|\|P\|\|\sum_{j=1}^{p} r_j \tilde{\epsilon}_{K_r}\| \end{aligned} \tag{2}$$

Note that as long as the matrix Ω is full ranked (which can be guaranteed if the reference signal r is exciting over a finite interval $[0, T]$ [1]) and $\texttt{sgn}(B) = \texttt{sgn}(\hat{B})$, the first three terms on the right hand side of the inequality above are negative-definite. Conservative bounds on the rest of the right hand side terms can be found as follows: The matrix A_{rm} of the reference model is assumed to be Hurwitz and the reference signal r is always bounded, therefore there exist scalars $c_r, c_{x_{rm}} > 0$ such that $\|x_{rm}\| < c_{rm}, \|r\| < c_r$. Note that $\|\tilde{B}\|$ is assumed to be bounded, therefore there exists a scalar c_B such that $\|P\|\|\tilde{B}\| < c_B$. Finally, note that the error terms $\hat{\epsilon}$ are functions of recorded data x_j, r_j, which are bounded by assumption and do not evolve with time. Therefore there exist scalars $c_{\epsilon_x}, c_{\epsilon_r} > 0$ such that $\|P\|\|\sum_{j=1}^{p} x_j \tilde{\epsilon}_K\| < c_{\epsilon_x}$, $\|P\|\|\sum_{j=1}^{p} r_j \tilde{\epsilon}_{K_r}\| < c_{\epsilon_r}$. Hence,

$$\begin{aligned} \dot{V}(\zeta) \;\; \leq \;\; & -\frac{1}{2}\lambda_{\min}(Q)\|e\|^2 - \frac{1}{2}\lambda_{\min}(\Omega)\|\tilde{K}\|^2 \\ & -\frac{1}{2}(\sum_{j=1}^{p} r_j^2)\|\tilde{K_r}\|^2 + c_{rm}c_B\|\tilde{K}\|\|e\| + c_B\|\tilde{K}\|\|e\|^2 \\ & +c_r c_B\|\tilde{K_r}\|\|e\| + c_{\epsilon_x}\|\tilde{K}\| + c_{\epsilon_r}\|\tilde{K_r}\|. \end{aligned} \quad (3)$$

Therefore, for sufficiently large $\lambda_{\min}(Q)$, $\lambda_{\min}(\Omega)$, and $\sum_{j=1}^{p} r_j^2$, $\dot{V}(\zeta) \leq 0$ outside of a compact set. To see that the set is indeed compact, note that the terms on the right hand side of Eq. 3 yield three quadratic inequalities in $\|e\|$, $\|\tilde{K}\|$ and $\|\tilde{K_r}\|$. A conservative estimate of the positively invariant set within which the solutions are bounded can be found by solving these quadratic inequalities for each variable while assuming that other two variables are non-zero. First we check the case where $\|\tilde{K}\| > 0, \|\tilde{K_r}\| > 0$. In this case,

$$\begin{aligned} \|e\| \;\; & \geq \;\; \frac{-b_e + \sqrt{b_e^2 - 4a_e c_e}}{2a_e} \\ a_e \;\; & = \;\; -\frac{1}{2}\lambda_{\min}(Q) + c_b\|\tilde{K}\| \\ b_e \;\; & = \;\; c_{rm}c_B\|\tilde{K}\| + c_r c_B\|\tilde{K_r}\| \\ c_e \;\; & = \;\; -\frac{1}{2}\lambda_{\min}(\Omega)\|\tilde{K}\|^2 - \frac{1}{2}(\sum_{j=1}^{p} r_j^2)\|\tilde{K_r}\|^2 \\ & \quad +c_{\epsilon_x}\|\tilde{K}\| + c_{\epsilon_r}\|\tilde{K_r}\| \end{aligned} \quad (4)$$

then $\dot{V}(\zeta) \leq 0$.

Secondly, we consider the case $\|e\| \geq 0, \|\tilde{K_r}\| \geq 0$. In this case, $\dot{V}(\zeta) \leq 0$ if

$$\begin{aligned} \|\tilde{K}\| \;\; & \geq \;\; \frac{-b_k + \sqrt{b_k^2 - 4a_k c_k}}{2a_k} \\ a_k \;\; & = \;\; -\frac{1}{2}\lambda_{\min}(\Omega), \\ b_k \;\; & = \;\; c_{rm}c_B\|e\| + c_B\|e\|^2 + c_{\epsilon_x} \\ c_k \;\; & = \;\; -\frac{1}{2}\lambda_{\min}(Q)\|e\|^2 - \frac{1}{2}(\sum_{j=1}^{p} r_j^2)\|\tilde{K_r}\|^2 \\ & \quad +c_r c_B\|\tilde{K_r}\|\|e\| + c_{\epsilon_r}\|\tilde{K_r}\| \end{aligned} \quad (5), (6)$$

Then we check the case where $\|e\| \geq 0, \|\tilde{K}\| \geq 0$.

$$\|\tilde{K_r}\| \geq \frac{-b_{k_r} + \sqrt{b_{k_r}^2 - 4a_{k_r}c_{k_r}}}{2a_{k_r}} \tag{7}$$

$$\begin{aligned} a_{k_r} &= -\frac{1}{2}(\sum_{j=1}^{p} r_j^2), \\ b_{k_r} &= c_r c_B \|e\| + c_{\epsilon_r} \\ c_{k_r} &= -\frac{1}{2}\lambda_{\min}(Q)\|e\|^2 - \frac{1}{2}\lambda_{\min}(\Omega)\|\tilde{K}\|^2 \\ & \quad c_{rm}c_B\|\tilde{K}\|\|e\| + c_B\|\tilde{K}\|\|e\|^2 + c_{\epsilon_x}\|\tilde{K}\|. \end{aligned} \tag{8}$$

Inequalities 4–7 characterize the compact set outside of which $\dot{V}(\zeta) \leq 0$. Therefore, all solutions will eventually end up within this set, which in turn proves that the system $[e, \tilde{K}, \tilde{K_r}]$ is uniformly ultimately bounded.

□

References

[1] G. Chowdhary, T. Yucelen, M. Mühlegg, and E. N. Johnson, "Concurrent learning adaptive control of linear systems with exponentially convergent bounds," *International Journal of Adaptive Control and Signal Processing*, 2012.