Parametric Robust $\mathcal{H}_2$ Control Design Using LMI Synthesis

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Abstract

This paper presents a new, iterative algorithm for designing full order LTI controllers for systems with real parameter uncertainty. Robust stability is determined for these systems using the Popov analysis criterion and multiplier, and robust performance is investigated using a bound on the output energy. Control design to minimize the robust performance metric naturally leads to Bilinear Matrix Inequalities, which can be decoupled to a large extent. However, coupling remains in the problem since we simultaneously optimize the parameters of both the Popov stability multiplier and the compensator. We present a heuristic, iterative algorithm to solve this design problem, and demonstrate that it works effectively on two numerical examples. In the process, we illustrate that the key advantages of this control design approach are the high reliability of the numerical techniques and the relative simplicity of implementing the algorithm.

1 Introduction

Control design to satisfy robust performance objectives with real parameter uncertainty has recently attracted much attention in the controls community. One of the many proposed techniques is Popov Controller Synthesis which captures the system uncertainties as sector-bounded nonlinearities [1]. Additional information about the type and structure of the system uncertainties is captured in the analysis using a frequency dependent Popov multiplier [2, 3] or its generalizations [4]. A Lyapunov function representation of the Popov analysis test can be combined with an $\mathcal{H}_2$ cost function to provide a bound on the robust performance. Designing a controller to optimize this bound provides a synthesis tool that guarantees robust stability and performance. Several previous investigations have been performed to develop solutions to this control design problem [1, 5, 6]. However, until recently, these approaches were based on numerical optimizations using quasi-Newton search algorithms to solve the necessary conditions which must be analytically derived from the performance objective and constraint equations. This previous work clearly demonstrated that Popov Controller Synthesis can be used to design very robust controllers, but there exist significant drawbacks to this approach in practice. These drawbacks include the significant computational effort required by the iterative gradient search algorithm, the difficulties of deriving the required gradients, and the difficulty of obtaining a good initial guess. The results in this paper introduce a new combined analysis and synthesis procedure that eliminates many of the numerical and implementation difficulties of the quasi-Newton approach, leading to efficient and effective robust control design technique for systems with linear and nonlinear parameter uncertainty.

This new approach is based on *Linear Matrix Inequalities* (LMI's), that Boyd *et al.* [7] have investigated for the Popov analysis problem. The direct extension of this LMI analysis to the controller synthesis problem results in *Bilinear Matrix Inequalities* (BMI's), which currently cannot be solved directly. El Ghaoui *et al.* [8] proposed a heuristic solution procedure for these BMI problems using a two-stage optimization process, called the *V-K iteration.* During each phase of this iteration, some of the design variables in the BMI's are fixed, leading to LMI's in the remaining solution variables. This technique has been shown to work well on simple examples [8], but

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on complicated objectives, such as control design to minimize an $\mathcal{H}_2$ cost function, has been found to converge very slowly, if at all. This synthesis algorithm was recently improved by El Ghaoui *et al.* [9] leading to a systematic design approach for systems with unstructured uncertainty.

The main result of this paper is to extend the results in Ref. [9] to develop synthesis algorithms for systems with parameter uncertainty. This objective is achieved by combining Popov analysis and robust performance bounds on the output energy of a nominal linear time invariant system subject to sector-bounded nonlinear uncertainties. We apply the LMI synthesis tools to solve this problem, and in the process show the difficulties that arise when we simultaneously select the optimal parameters for both the multiplier and the compensator. An extension of the procedure in Ref. [9] is then presented to overcome these difficulties in our resulting BMI problem. Note that a similar approach to this problem is also discussed by Yang *et al.* [10].

The paper is organized as following. In next section, we define the problem statement in detail. Section 3 describes the sufficient conditions for the existence of the robust controllers, followed by the design specifications in Section 4. Section 5 then briefly describes the elimination and completion lemmas which are used in controller synthesis. The design procedure is given in Section 6 followed by numerical examples in Section 7.

2 Problem Statement

We consider an LTI system, *i.e.*, the nominal system, subject to sector bounded nonlinearities, *i.e.*, a Lur'e system described by

$$\begin{array}{rcl} \dot{x} &=& Ax + B_p p + B_w w + B_u u \\ q &=& C_q x + D_{qp} p + D_{qw} w + D_{qu} u \\ z &=& C_z x + D_{zp} p + D_{zw} w + D_{zu} u \\ y &=& C_y x + D_{yp} p + D_{yw} w + D_{yu} u \\ p_i &=& \phi_i(q_i),\ i = 1, \ldots, n_p \end{array} \qquad (1)$$

where $x : \mathbf{R}_+ \to \mathbf{R}^n$ is the state, $u : \mathbf{R}_+ \to \mathbf{R}^{n_u}$ is the control input, $w : \mathbf{R}_+ \to \mathbf{R}^{n_w}$ is the disturbance input, $y : \mathbf{R}_+ \to \mathbf{R}^{n_y}$ is the measured output and $z : \mathbf{R}_+ \to \mathbf{R}^{n_z}$ is the output. The perturbations ϕ_i are nonlinearities that satisfy the sector bound $[0, 1]$, *i.e.*, $0 \le \phi_i(\sigma)/\sigma \le 1$. As discussed in Refs. [11, 7], we can use a loop transformation to handle the more general sector conditions $\alpha_i \le \phi_i(\sigma)/\sigma \le \beta_i$. Our goal is to find a strictly proper full order LTI controller, of the form

$$\begin{array}{rcl} \dot{x}_c &=& A_c x_c + B_c y \\ u &=& C_c x_c \end{array}$$

where $x_c : \mathbf{R}_+ \to \mathbf{R}^n$ is the controller state, and A_c, B_c and C_c are constant matrices of appropriate size such that some desirable specifications, defined in Section 4, are achieved. The closed loop system of the Lur'e system and the LTI controller is described by

$$\begin{array}{rcl} \dot{\tilde{x}} &=& \tilde{A}\tilde{x} + \tilde{B}_p p + \tilde{B}_w w \\ q &=& \tilde{C}_q \tilde{x} + \tilde{D}_{qp} p + \tilde{D}_{qw} w \\ z &=& \tilde{C}_z \tilde{x} + \tilde{D}_{zp} p + \tilde{D}_{zw} w \\ p_i &=& \phi_i(q_i),\ i = 1, \ldots, n_p \end{array} \qquad (2)$$

where $\tilde{x} = [\ x^T \quad x_c^T\ ]^T$ and

$$\left[\begin{array}{c|c|c} \tilde{A} & \tilde{B}_p & \tilde{B}_w \\ \hline \tilde{C}_q & \tilde{D}_{qp} & \tilde{D}_{qw} \\ \hline \tilde{C}_z & \tilde{D}_{zp} & \tilde{D}_{zw} \end{array}\right] = \left[\begin{array}{cc|c|c} A & B_u C_c & B_p & B_w \\ B_c C_y & A_c + B_c D_{yu} C_c & B_c D_{yp} & B_c D_{yw} \\ \hline C_q & D_{qu} C_c & D_{qp} & D_{qw} \\ \hline C_z & D_{zu} C_c & D_{zp} & D_{zw} \end{array}\right]$$

For well-posedness, we will assume that D_{zw} is identically zero, and to significantly simplify the analysis and synthesis, we assume D_{zp}, D_{qp}, D_{qw}, and D_{qu} are identically zero. In next section, we present the robust performance analysis and synthesis of the closed loop Lur'e system (2).

3 Popov Analysis/Synthesis

The Popov analysis is based on Lyapunov functions of the form

$$V(\tilde{x}) = \tilde{x}^T \tilde{P} \tilde{x} + 2 \sum_{i=1}^{n_p} \lambda_i \int_0^{\tilde{C}_{i,q}\tilde{x}} \phi_i(\sigma)\, d\sigma \qquad (3)$$

where $\tilde{C}_{i,q}$ denotes the i^{th} row of $\tilde{C}_q$. Thus the data describing the Lyapunov function are the matrix $\tilde{P}$ and the scalars λ_i, $i = 1, \ldots, n_p$. We require $\tilde{P} > 0$ and $\lambda_i \ge 0$, which implies that $V(\tilde{x}) \ge \tilde{x}^T \tilde{P} \tilde{x} > 0$ for nonzero $\tilde{x}$. Let $w = 0$, $\tilde{x}(0)$ be the initial condition of the closed loop system, we will compute an upper bound on the output energy

$$J_{\tilde{x}(0)} = \int_0^\infty z^T z\, dt$$

using Lyapunov functions V of the form in q. (3). If

$$\frac{d}{dt} V(\tilde{x}) + z^T z \le 0,\ \forall \tilde{x} \text{ satisfying Eq. (2)}, \qquad (4)$$

then $J_{\tilde{x}(0)} \le V(\tilde{x}(0))$. The condition (4) is equivalent to

$$2 \left(\tilde{x}^T \tilde{P} + \sum_{i=1}^{n_p} \lambda_i \pi_i \tilde{C}_{i,q} \right) \left(\tilde{A}\tilde{x} + \tilde{B}_p \pi \right) + \tilde{x}^T \tilde{C}_z^T \tilde{C}_z \tilde{x} \le 0,$$

for every $\tilde{x}$ satisfying

$$\pi_i(\pi_i - \tilde{C}_{i,q}\tilde{x}) \leq 0, \; i = 1, \dots, n_p.$$

Using the $\mathcal{S}$-procedure [7], we can conclude that condition (4) holds if there exists $T = \mathbf{diag}(\tau_1, \dots, \tau_{n_p}) \geq 0$ such that

$$\begin{bmatrix} \tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} & \tilde{P}\tilde{B}_p + \tilde{A}^T\tilde{C}_q^T\Lambda \\ + \tilde{C}_z^T\tilde{C}_z & + \tilde{C}_q^T T \\ \tilde{B}_p^T\tilde{P} + \Lambda\tilde{C}_q\tilde{A} & \Lambda\tilde{C}_q\tilde{B}_p + \tilde{B}_p^T\tilde{C}_q^T\Lambda \\ + T\tilde{C}_q & - 2T \end{bmatrix} \leq 0, \quad (5)$$

where $\Lambda = \mathbf{diag}(\lambda_1, \dots, \lambda_{n_p})$. Since $V(\tilde{x}(0)) \leq \tilde{x}(0)^T[\tilde{P} + \tilde{C}_q^T\Lambda\tilde{C}_q]\tilde{x}(0)$, an upper bound on $J_{\tilde{x}(0)}$ is obtained by solving the following optimization problem in the variables $\tilde{P}$, A_c, B_c, C_c, Λ and T:

$$\begin{array}{ll} \text{minimize} & \tilde{x}(0)^T\left[\tilde{P} + \tilde{C}_q^T\Lambda\tilde{C}_q\right]\tilde{x}(0) \\ \text{subject to} & (5),\; \tilde{P} > 0,\; T \geq 0,\; \Lambda \geq 0. \end{array} \quad (6)$$

4 Design Specifications

We now consider the specific form of the cost overbound in (6) for a linear system with nonlinear sector-bounded uncertainties. Our design specifications are:

(1) Robust Stability: Let $w = 0$, for $t \geq 0$ all trajectories of the state response $\tilde{x}(t)$ go to zero.

(2) Robust Performance: Consider an LTI system $G \sim \{A, B_w, C_z, 0\}$, where A, B_w, C_z have the dimensions defined earlier. The following definitions for $\mathcal{H}_2$ norm of the system G are equivalent [12, 13].

  (a) Let $e_1, \dots, e_{n_w}$ be a basis of the input space (*i.e.*, the i^{th} element of e_i equals to 1 and the rest are zero). Let $z_i(t)$ be the output of the system if we apply the impulse δe_i to the system and set the initial condition to be zero. Then the $\mathcal{H}_2$ norm of the system G is defined as:

$$\|G\|_2^2 \triangleq \sum_{i=1}^{n_w} \|z_i\|_2^2$$

  where $\|z_i\|_2$ denotes the $\mathcal{L}_2$ norm of z_i.

  (b) Let $x_{0,i} = B_w e_i, i = 1, \dots, n_w$ be the initial conditions of the system G. Let $z_{0,i}$ denote the output response subject to initial condition $x_{0,i}$ and $w = 0$. Then the $\mathcal{H}_2$ norm of the system G is defined as:

$$\|G\|_2^2 \triangleq \sum_{i=1}^{n_w} \|z_{0,i}\|_2^2.$$

The calculation of the $\mathcal{H}_2$ norm is straightforward. If A is stable, $\|G\|_2^2 = \mathbf{Tr} B_w^T P B_w$ where P is a unique positive symmetric matrix satisfying:

$$A^T P + PA + C_z^T C_z = 0.$$

In case of nonlinear systems, it is no longer relevant to discuss the performance in terms of an $\mathcal{H}_2$ norm. However, as discussed in Ref. [12], it is still valid to use the second definition.

With these results, the performance of the closed-loop system in Eq. (2) is defined as follows. Let $w = 0$, $\tilde{x}_{0,i}, i = 1, \dots, n_w$ be a basis of the input space defined by $\tilde{x}_{0,i} = \tilde{B}_w e_i$, and define a bound on the output energy, J_i, as a result of the initial condition $\tilde{x}_{0,i}$

$$J_i \leq \tilde{B}_{w,i}^T\left(\tilde{P} + \tilde{C}_q^T\Lambda\tilde{C}_q\right)\tilde{B}_{w,i}$$

where $\tilde{B}_{w,i}$ denotes the i^{th} column of $\tilde{B}_w$. In this case, the overall cost $J \triangleq \sum_{i=1}^{n_w} J_i$ is bounded by

$$J \leq \mathbf{Tr}\tilde{B}_w^T\left(\tilde{P} + \tilde{C}_q^T\Lambda\tilde{C}_q\right)\tilde{B}_w \quad (7)$$

The robust performance design specification is that

$$\mathbf{Tr}\tilde{B}_w^T\left(\tilde{P} + \tilde{C}_q^T\Lambda\tilde{C}_q\right)\tilde{B}_w \leq \gamma^2 \quad (8)$$

where $\gamma > 0$ is a pre-specified number.

In summary, the robust $\mathcal{H}_2$ performance problem is equivalent to the following optimization problem:

$$\begin{array}{ll} \text{minimize} & \gamma^2 \\ \text{subject to} & (5),\; (8),\; \tilde{P} > 0,\; T \geq 0,\; \Lambda \geq 0. \end{array} \quad (9)$$

Note that when the compensator parameters (A_c, B_c, C_c) are fixed, (9) results in the Popov analysis problem.

Remark: Up to this point, the formulation of the optimal Popov Controller Synthesis problem is virtually identical to previous work since we can take the Schur complement of (5) to develop the Riccati equations used in that work. For numerical optimization techniques, the cost overbound in (7) is augmented with the constraint equations for the robust analysis test to develop the Lagrangian. Analytic gradients this Lagrangian are then computed to determine the necessary conditions that can be solved via a quasi-Newton optimization. A key advantage of this numerical optimization approach is that constraints on the compensator order and architecture can very easily be included into the optimization. However, some of the difficulties involved with this approach are the substantial effort required to compute the analytic gradients, the difficulties of initializing the algorithm, and the slow rate of convergence for large order problems [1]. In the following sections, we extend the work in Ref. [9] to develop an LMI synthesis algorithm for solving this robust performance problem. As will be shown, controllers are developed in two main steps. The first step does require an iteration, but in the process capitalizes on the very efficient design tools that are available for solving LMI problems [14, 15]. The resulting compensators are full-order, and cannot include architecture constraints. However, the solution procedure is very robust which reduces the user workload, and, is easily expandable to include other analysis tests.

5 Preliminaries

The following well-known lemmas will be very useful in developing the controller design technique in Section 6.

Lemma 1 [Elimination Lemma] *Let* $G \in \mathbf{R}^{n\times n}$, $U \in \mathbf{R}^{n\times p}$ *and* $V \in \mathbf{R}^{n\times q}$. *We define* $U_\perp$ *be an orthogonal complement of* U, *i.e.,* $U^T U_\perp = 0$ *and* $[U\ U_\perp]$ *is of maximum rank. Similarly,* $V_\perp$ *is defined as an orthogonal complement of* V. *There exists a matrix* $X \in \mathbf{R}^{p\times q}$ *such that*

$$G + VX^TU^T + UXV^T < 0,$$

if and only if

$$V_\perp^T G V_\perp < 0,\ U_\perp^T G U_\perp < 0.$$

Proof: *See Ref. [7], (page 32).*

Lemma 2 [Completion Lemma] *Let* P *and* $Q \in \mathbf{R}^{n\times n}$ *be two positive matrices. There exists a positive matrix* $\tilde{P} \in \mathbf{R}^{2n\times 2n}$ *such that the upper left* $n\times n$ *block of* $\tilde{P}$ *is* P, *and that of* $\tilde{P}^{-1}$ *is* Q, *if and only if*

$$\begin{bmatrix} P & I \\ I & Q \end{bmatrix} \geq 0. \qquad (10)$$

Proof: *See Ref. [16].*

For each pair of matrices P and Q that strictly satisfy (10), the set of matrices $\tilde{P}$ satisfying the conditions in Lemma 2 is parameterized by

$$\tilde{P} = \begin{bmatrix} I & 0 \\ 0 & M^T \end{bmatrix} \begin{bmatrix} P & I \\ I & (P-Q^{-1})^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, \qquad (11)$$

where $M \in \mathbf{R}^{n\times n}$ is an arbitrary invertible matrix. Then $\tilde{Q} \triangleq \tilde{P}^{-1}$, *i.e.,*

$$\tilde{Q} = \begin{bmatrix} I & 0 \\ 0 & N^T \end{bmatrix} \begin{bmatrix} Q & I \\ I & (Q-P^{-1})^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \qquad (12)$$

where $N = (I - QP)M^{-1}$.

6 Controller Synthesis

The following sections very closely parallel the developments in Ref. [9] which considers systems with unstructured uncertainty. The main point in the parametric uncertainty case presented here is the difference in the amount of decoupling in the problem variables that is achieved using the elimination lemma.

6.1 Controller Elimination

We first note that the controller matrix A_c only appears in the inequality (5). Thus it is possible to reduce the number of variables in the problem by eliminating A_c. To proceed, we define

$$\tilde{A}_0 \triangleq \begin{bmatrix} A & B_uC_c \\ B_cC_y & B_cD_{yu}C_c \end{bmatrix},$$

$$\tilde{J} \triangleq \begin{bmatrix} 0 \\ I \end{bmatrix},\quad \tilde{J}_\perp \triangleq \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Then $\tilde{A}$ can be written as $\tilde{A} = \tilde{A}_0 + \tilde{J}A_c\tilde{J}^T$ and we rewrite inequality (5) as

$$\tilde{G} + \begin{bmatrix} \tilde{J} \\ 0 \end{bmatrix} A_c^T \begin{bmatrix} \tilde{P}\tilde{J} \\ 0 \end{bmatrix}^T + \begin{bmatrix} \tilde{P}\tilde{J} \\ 0 \end{bmatrix} A_c \begin{bmatrix} \tilde{J} \\ 0 \end{bmatrix}^T < 0. \qquad (13)$$

where

$$\tilde{G} = \begin{bmatrix} \tilde{A}_0^T\tilde{P} + \tilde{P}\tilde{A}_0 + \tilde{C}_z^T\tilde{C}_z & \tilde{P}\tilde{B}_p + \tilde{A}_0^T\tilde{C}_q^T\Lambda + \tilde{C}_q^T T \\ \tilde{B}_p^T\tilde{P} + \Lambda\tilde{C}_q\tilde{A}_0 + T\tilde{C}_q & \Lambda\tilde{C}_q\tilde{B}_p + \tilde{B}_p^T\tilde{C}_q^T\Lambda - 2T \end{bmatrix}.$$

Here U and V are defined as

$$V = \begin{bmatrix} \tilde{J} \\ 0 \end{bmatrix},\quad U = \begin{bmatrix} \tilde{P}\tilde{J} \\ 0 \end{bmatrix}.$$

Therefore, their complements are

$$V_\perp = \begin{bmatrix} \tilde{J}_\perp & 0 \\ 0 & I \end{bmatrix},\quad U_\perp = \begin{bmatrix} \tilde{P}^{-1}\tilde{J}_\perp & 0 \\ 0 & I \end{bmatrix}.$$

Applying the elimination lemma, it follows that inequality (5) holds if and only if

$$\left[\begin{array}{cc} \tilde{J}_\perp & 0 \\ 0 & I \end{array}\right]^T \tilde{G} \left[\begin{array}{cc} \tilde{J}_\perp & 0 \\ 0 & I \end{array}\right] < 0,$$

$$\left[\begin{array}{cc} \tilde{P}^{-1}\tilde{J}_\perp & 0 \\ 0 & I \end{array}\right]^T \tilde{G} \left[\begin{array}{cc} \tilde{P}^{-1}\tilde{J}_\perp & 0 \\ 0 & I \end{array}\right] < 0.$$

To proceed, we partition $\tilde{P}$ and its inverse $\tilde{Q}$ as

$$\tilde{P} = \left[\begin{array}{cc} P & M \\ M^T & R \end{array}\right], \quad \tilde{Q} = \left[\begin{array}{cc} Q & N \\ N^T & S \end{array}\right]. \qquad (14)$$

where P and $Q \in \mathbf{R}^{n \times n}$. Then, after some algebra, it can be shown that the matrix inequality (5) is equivalent to

$$\left[\begin{array}{cc} F_{11} & F_{12} \\ F_{12}^T & F_{22} \end{array}\right] < 0, \quad \left[\begin{array}{ccc} H_{11} & H_{12} & H_{13} \\ H_{12}^T & H_{22} & 0 \\ H_{13}^T & 0 & -I \end{array}\right] < 0, \qquad (15)$$

where

$$\begin{array}{rcl} F_{11} & = & PA + A^T P + ZC_y + C_y^T Z^T + C_z^T C_z, \\ F_{12} & = & PB_p + ZD_{yp} + A^T C_q^T \Lambda + C_q^T T, \\ F_{22} & = & \Lambda C_q B_p + B_p^T C_q^T \Lambda - 2T, \\ H_{11} & = & AQ + Q^T A^T + B_u Y + Y^T B_u^T, \\ H_{12} & = & B_p + (Q^T A + Y^T B_u^T) C_q^T \Lambda + QC_q^T T, \\ H_{13} & = & Q^T C_z^T + Y^T D_{zu}^T, \\ H_{22} & = & \Lambda C_q B_p + B_p^T C_q^T \Lambda - 2T, \end{array}$$

where Y and Z are defined as:

$$Y \triangleq C_c N^T, \; Z \triangleq MB_c. \qquad (16)$$

By the completion lemma, the conditions $\tilde{P} > 0$ and $\tilde{P}\tilde{Q} = I$ with $\tilde{P}$ given by (14) implies the inequality (10). By relaxing inequality (10) to be strict, *i.e.*,

$$\left[\begin{array}{cc} P & I \\ I & Q \end{array}\right] > 0, \qquad (17)$$

we are effectively searching for full-order controllers (*i.e.*, of order n) [9].

We observe that the second inequality in (15) is Bilinear Matrix Inequality (BMI), *i.e.*, there are product terms involving (Q, Y) and (Λ, T). This is a direct consequence of optimizing simultaneously both the compensator parameters (related to Q and Y) and the analysis multiplier (Λ, T). Note that if (Λ, T) are fixed, then the inequalities (15) are Linear Matrix Inequalities (LMI's) in (Q, Y). Similarly, if (Q, Y) are fixed, then the inequalities (15) are LMI's in (Λ, T).

Now consider the condition of the energy bound appeared in the robust performance specification (8). The trace condition is handled by noting that it is equivalent to the existence of a symmetric matrix V such that

$$\mathbf{Tr} \left[\begin{array}{c} B_w \\ D_{yw} \end{array}\right]^T \left[\begin{array}{cc} P + C_q^T \Lambda C_q & Z \\ Z^T & V \end{array}\right] \left[\begin{array}{c} B_w \\ D_{yw} \end{array}\right] < \gamma^2,$$

$$\left[\begin{array}{ccc} V & Z^T & 0 \\ Z & P & I \\ 0 & I & Q \end{array}\right] > 0. \qquad (18)$$

We note that the last inequality of (18) implies (17). In summary, after eliminating A_c from the formulation, the robust performance problem (9) is equivalent to:

$$\begin{array}{ll} \text{minimize} & \gamma^2 \\ \text{subject to} & (15),\ (18),\ T \geq 0,\ \Lambda \geq 0 \end{array} \qquad (19)$$

6.2 Controller Reconstruction

Given that there exist P, Q, Y, Z, V, Λ and T satisfying (19), we can construct a controller as follows. First we construct the Lyapunov function, *i.e.*, $\tilde{P}$, such that the condition (5) holds. The set of closed-loop Lyapunov functions is parameterized by (11), where M is an arbitrary invertible matrix. Because M corresponds to a change of coordinates in the controller states x_c, the choice of M has no effect on the controller transfer function [9].

After constructing the Lyapunov function, the set of input/output controller matrices (B_c and C_c) can be parameterized by (16). With $\tilde{P}, \Lambda, T, B_c$, and C_c determined, it suffices to find A_c satisfying the condition (13), which can then be formulated as an LMI problem in A_c.

6.3 Algorithm

The solution algorithm discussed above is summarized below:

(1) Initialize the sector bound nonlinearity to be zero (a nominal system) and design the controller via Linear Quadratic Gaussian (LQG) or any other robust technique.

(2) Initialize Λ and T by Popov analysis.

(3) repeat{ [Outer Loop]

repeat{ [Inner Loop]

i. Solve the optimization problem (19), *i.e.*, alternate between solving for (P, Z, Q, Y) and (P, Z, Λ, T). Then compute $\tilde{P}, B_c$, and C_c using the completion lemma.

ii. Compute A_c by solving a feasibility problem (13).
iii. Compute Λ and T by Popov analysis.

} [Inner Loop] until stopping criterion satisfied.

Increase the sector bound nonlinearity to the next desired size and initialize Λ and T by the most recent values.

} [Outer Loop] until the desired robustness is achieved or the problem is infeasible.

Remark 1: The procedure of alternating between the LMI problems is a heuristic approach of solving a non-convex optimization problem. It is guaranteed to converge, but not necessarily to the global optimum [8]. However, each step of the iteration can be solved very efficiently by a previously developed semidefinite programming algorithm *sp* [14] and very easily coded using a user-friendly interface *sdpsol* [15]. The simplicity of this *sdpsol* interface is demonstrated in Table 1 which shows the code required to solve (3iii) of the numerical algorithm. The problem statements are written in a file called `popov.sdp` which is shown in Table 1. The system parameters A, B_p, C_q, B_w and C_z are written from MATLAB into the file called `SystemsPar.mat`. Note that we omit the $\tilde{(\cdot)}$ in the parameter notation. The *sdpsol* program then generates the semidefinite programming problem which is automatically solved by calling the optimization engine *sp*. The parameters used in *sp* are stored in `SPPar.mat`. After completion of the optimization, *sdpsol* writes the solution into a MATLAB readable file.

Remark 2: Note that the solution variables (P, Z) are shared between the two stages of the iteration, and we conjecture that this sharing plays an important role in the reliability of this heuristic algorithm for solving the control design problem.

7 Numerical Examples

One of the first objectives with a new synthesis tool such as the one presented in this paper is to confirm that the approach is consistent with previous results. We confirm the designs from this algorithm by comparing our controllers to those obtained for two benchmark problems. The first benchmark problem was selected from Ref. [6] as an external check of the synthesis algorithm itself. The second benchmark problem from Ref. [17] was selected to enable a more thorough comparison of robust Popov controllers with various other robust $\mathcal{H}_2$ techniques.

Table 1: Typical *sdpsol* program for Popov analysis. The simplicity of this user interface significantly reduces the overhead associated with developing and implementing the Popov control synthesis algorithm.

```
% popov.sdp:
% Popov analysis for the Lur'e system

% SystemPar.mat stores system parameters:
% A, Bp, Cq, Bw, Cz
include("SystemPar.mat")

% SPPar.mat stores SP parameters:
% ABSTOL, NU, MAXITER, BIGM
include("SPPar.mat")
n = rows(A); nq = rows(Cq);

variable P(n,n) symmetric;
variable L(nq,nq), T(nq,nq) diagonal;

% cost function
minimize Bw'*(P+Cq'*L*Cq)*Bw;

% LMI constraints
[A'*P+P*A+Cz'*Cz, P*Bp+A'*Cq'*L+Cq'*T;
Bp'*P+L*Cq*A+T*Cq, L*Cq*Bp+Bp'*Cq'*L-2*T] < 0;
P > 0; L > 0; T > 0;
```

7.1 Three-Mass System

The first system is taken from the benchmark problem described in Ref. [6]. The plant consists of three masses connected by springs, as illustrated in Fig. 1, where $m_1 = m_2 = m_3 = 1$, and $k_1 = 1$. The spring uncertainty between the second and third masses is written as $k_2 = k_{2,nom} + \delta$, where $k_{2,nom} = 1$ is the nominal value and the uncertainty is captured by $\delta \in \mathbf{R}$.

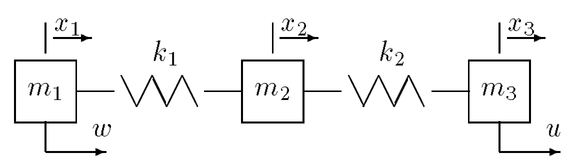

Figure 1: Three spring mass system. The second spring k_2 is treated as the uncertain parameter.

To apply this Popov analysis and synthesis, we effectively approximate the uncertainty in the spring stiffness as $k_2(x) = k_{2,nom}(x + \sigma\phi(x))$, where $\sigma > 0$ is a measure of the relative uncertainty size, and $\phi(x)$ is a $[-1, 1]$ sector-bounded memoryless nonlinear function of the spring displacement, x. Thus this approach can be used to treat sector-bounded nonlinear uncertainties or approximate linear uncertainties.

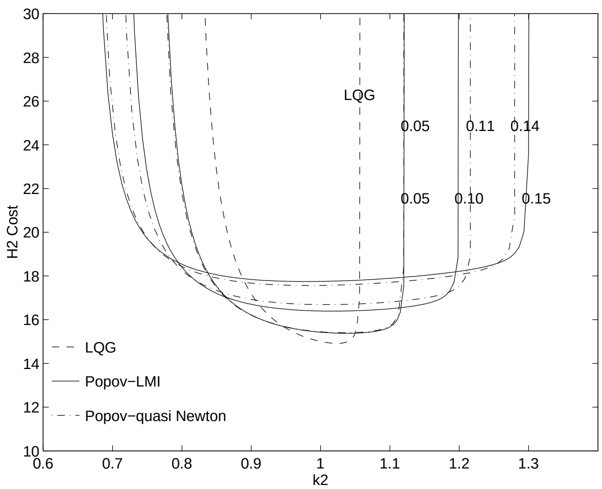

Figure 2: Robust performance comparison of Popov controllers designed using LMI synthesis and quasi-Newton methods [6] (Note in that paper $\gamma = 1/\sigma$).

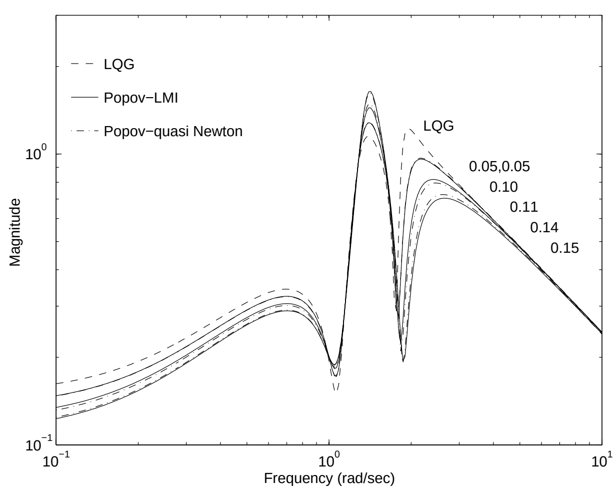

Figure 3: Frequency response of the seven controllers for the three-mass problem. The two different approaches generate very consistent results.

Several controllers were designed using the LMI synthesis algorithm presented in Section 6 for various values of σ. The robust controllers in Ref. [6] are tabulated in the Appendix of that paper, so these can be used to directly compare our compensators to those obtained via quasi-Newton methods.

The robust performance results are presented in three graphs. The first, in Fig. 2 is a standard plot which compares the $\mathcal{H}_2$ cost of the uncertain system as a function of changes in the spring stiffness k_2. The vertical asymptotes in Fig. 2 correspond to the stability boundaries for each compensator. Seven controllers are compared in the figure: one reference LQG design, three Popov designs from Ref. [6] designed at $\sigma = [\ 0.05,\ \ 0.11,\ \ 0.14\ ]$, and three Popov designs designed using LMI synthesis at $\sigma = [\ 0.05,\ \ 0.10,\ \ 0.15\ ]$.

First of all, note that, as expected, the Popov designs are more robust than the LQG controller, and that Fig. 2 clearly shows the trade-off between performance and robustness. However, most importantly for this study, note that the two Popov controllers with $\sigma = 0.05$, which were designed using two completely independent methods, give almost exactly the same robust performance curves. The other four controllers were designed for differing values of σ, and Fig. 2 shows that they exhibit consistent increases in the nominal performance and stability bounds.

To continue this analysis, we compare the frequency response of the seven controllers in Fig. 3 and the s-plane location of the compensator poles and zeroes in Fig. 4. Both plots confirm that the two designs with $\sigma = 0.05$ are essentially identical, and that the controllers for larger values of σ show consistent changes to achieve robustness. These results demonstrate the robustness of Popov controllers, but also confirm, through an external check, that the synthesis algorithm presented in this paper gives consistent control designs.

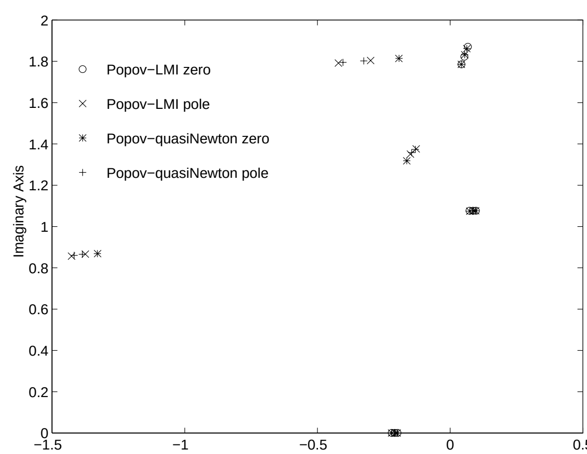

Figure 4: Poles and zeroes of the Popov controllers. The comparison shows that the results with $\sigma = 0.05$ are essentially identical.

7.2 Bernoulli Euler Beam

To further explore the robustness properties of these Popov controllers, we turn to the very detailed analysis of robust $\mathcal{H}_2$ synthesis techniques performed by Grocott *et al.* [17]. In their paper, the authors compare several robust control design techniques using benchmark problems based on a cantilevered Bernoulli Euler beam with unit length and mass density, and stiffness scaled so that the fundamental frequency is 1 rad/sec. The infinite order dynamics of the beam are truncated at four modes, where $w_1 = 1$ rad/sec, $w_2 = 6.27$ rad/sec, $w_3 = 17.55$ rad/sec, $w_4 = 34.39$ rad/sec and damping $\zeta = 0.01$. The disturbance input, control input, sensor output and performance output are all collocated at the tip of the beam, and the frequency of the third mode of the system is considered to be uncertain. The changes in the system dynamics due to perturbations in the frequency of the third mode are shown in the frequency response from the actuator to the sensor in Fig. 5. With $\pm 5\%$ shifts in the mode frequency, the plots shows substantial variations in the system gain and phase in the $17 - 25$ rad/sec frequency range.

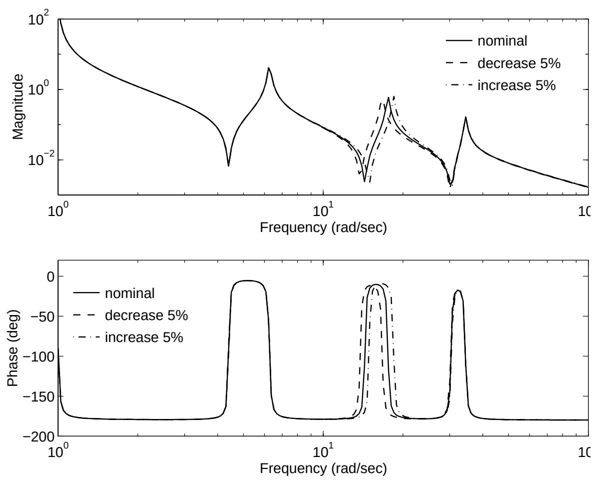

Figure 5: Changes in the beam dynamics caused by $\pm 5\%$ variations in the frequency of the third mode [17].

As with the previous example, several controllers were designed using the LMI synthesis approach (using sector increments of 1% to a limit of 10%). Note that for this sample problem, each iteration of the outer-loop in the algorithm of Section 6.3 required approximately 4 minutes to execute on a Sun-Sparc 20/60. With this reliable design technique, is now feasible to undertake a comparison of the Popov controllers with other robust $\mathcal{H}_2$ design techniques investigated by Grocott *et al.* [17]. Note that the Popov controller synthesis technique provides robust performance guarantees for (non)linear parameter variations within the entire sector bound, which separates this approach from other $\mathcal{H}_2$ synthesis techniques discussed in Ref. [17].

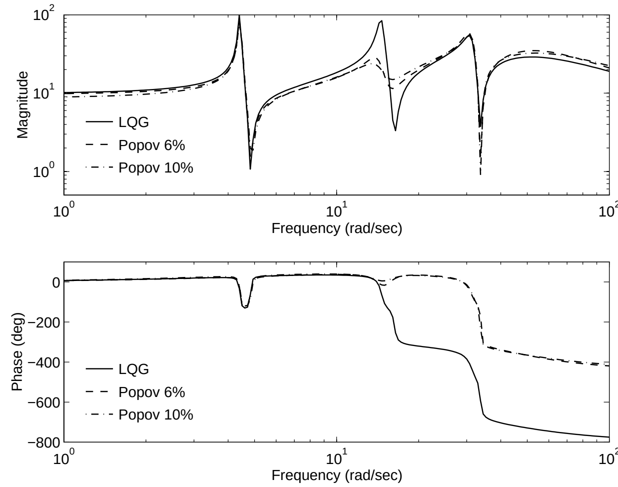

Figure 6: Frequency response of controllers robustified to changes in the frequency of the third mode. Significant changes to the response are apparent in the $17 - 25$ rad/sec range. The gains also change at high and low frequency.

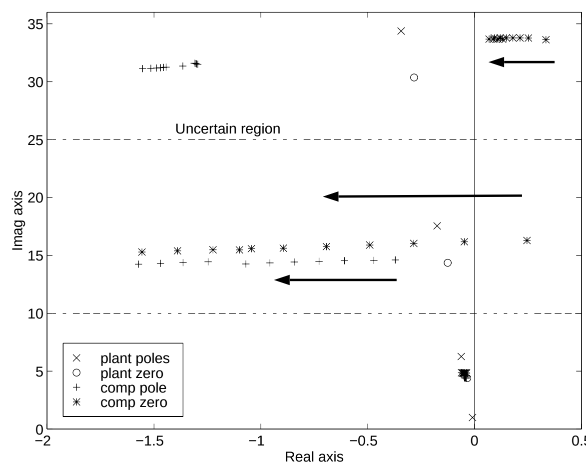

Figure 7: Poles and zeroes of the Popov controllers. The arrows show the direction of change with increasing robustness. Large changes in the uncertain region are clearly evident.

The frequency responses of three Popov compensators are compared with the reference LQG controller in Fig. 6. As discussed in Ref. [17], the other robust $\mathcal{H}_2$ designs change significantly in the 17–25 rad/sec range, and other secondary effects occur at higher and lower frequencies. Fig. 6 shows similar changes in the frequency responses of the Popov controllers. The large changes in this frequency range are even more apparent in the compensator pole-

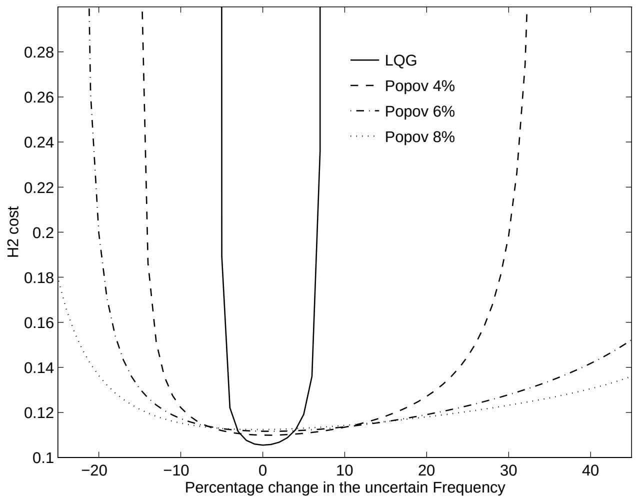

Figure 8: Robust performance plots for Popov controllers designed with the symmetric sector bounds given in the legend.

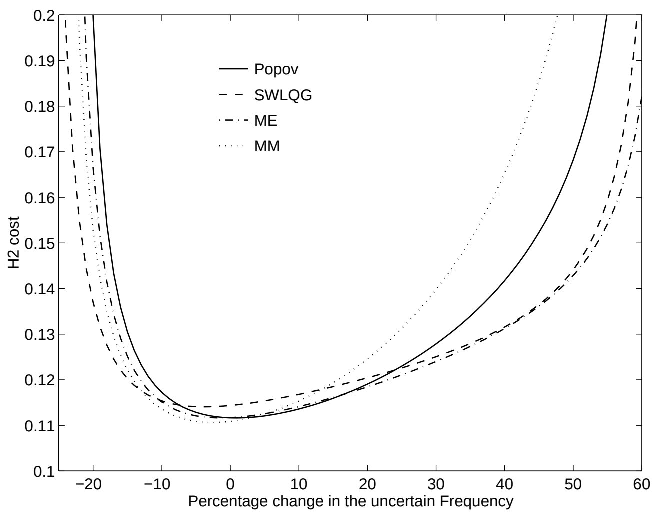

Figure 9: Comparison of performance plots for four robust $\mathcal{H}_2$ controller techniques that achieve similar levels of actual robustness.

zero plot in Fig. 7. This plot shows quite dramatically that, with increasing robustness, the compensator pole and zero in the uncertain region (17-25 rad/sec) both move further into the left-hand plane. In the process, the compensator zero near the frequency of the uncertain pole actually changes from non-minimum phase to minimum phase. The fact that the pole and zero move closer together in frequency and become more heavily damped is also consistent with the lower compensator gain in this frequency range shown in Fig. 7. Fig. 6 also shows that as the levels of robustness are increased, the compensators undergo a gain reduction at low frequency and a gain increase at high frequency, which are consistent with the changes shown in Ref. [17] for other robust $\mathcal{H}_2$ techniques.

The performance plots for three of the Popov controllers are shown in Fig. 8. As before, the curves in the figure are computed by calculating the $\mathcal{H}_2$ cost for the system with the given percentage change in the frequency of the mode. These controllers were designed using symmetric sector bounds, with the sizes (*i.e.*, $\pm 4\%$) given in the figure legend. Fig. 8 shows that the actual robust performance is quite asymmetric about the nominal stiffness. This asymmetry was one of the characteristics of Popov controllers identified during some of the earlier results in Refs. [5, 18], and indicates that asymmetric sector bounds might further reduce conservatism. We continue this discussion by directly comparing controllers designed four different ways: Sensitivity weight LQG (SWLQG) [17], Maximum Entropy (ME) [19], Multiple Model (MM) [20], and Popov controller synthesis (Popov) using LMI synthesis. Fig. 9 compares the robust performance of three controllers (analyzed in detail in Ref. [17]) that achieve approximately the same levels of robust stability as one of the Popov designs (which guarantees $\pm 6\%$). The plot shows that the actual performance curves for the four controllers are quite similar in terms of the stability boundaries obtained (in particular for the more critical lefthand side corresponding to a reduction in modal frequency). The ME, MM, and Popov curves also show similar cost increases for the nominal system dynamics. However, as discussed earlier, the Popov controller has the additional advantage that it provides a certificate of guaranteed robust performance for $\pm 6\%$ variations in the frequency of the mode.

8 Conclusions

This paper presents an iterative technique for Popov controller design using LMI synthesis. This approach was shown to agree with previously published work based on quasi-Newton numerical optimization. A second numerical example showed that the changes to the Popov controllers are consistent with other robust $\mathcal{H}_2$ designs. For these problems, the certificate of guaranteed robust performance associated with Popov controller synthesis is achieved without leading to overly conservative designs. In comparison to gradient optimization solution techniques, one of the significant advantages of LMI synthesis is the low overhead associated with developing and implementing the optimization conditions. This advantage will simplify the extension of this work to include new robust stability tests based on more generalized multipliers.

Acknowledgements

The authors would like to thank S. Grocott at MIT for his help comparing the Popov controllers to the other control designs. We would also like to thank L. El Ghaoui and J. Folcher for providing us with a preprint of Ref. [9]. This research was supported by Ananda Mahidol Foundation and in part by AFOSR (under F49620-95-1-0318).

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